universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„On the possible interactions in higher-spin gauge theory“

verfasst von / submitted by

David C. Vergeiner, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2021 / Vienna, 2021

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 876
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Physik
Betreut von / Supervisor:	Univ.-Prof. Dipl.-Phys. Dr. Stefan Fredenhagen
Mitbetreut von / Co-Supervisor:	

ABSTRACT

We investigate the possible perturbative interactions in the Lagrangian formulation of higher-spin gauge theories. Considering higher-spin theories of arbitrary integer spin in Minkowski spacetime of arbitrary dimension and working in the metric-like approach, we employ BRST-BV cohomological methods to work towards determining all consistent, local, Poincaré and parity invariant, first-order deformations of the free theory. We give a rough argument as to why certain cohomology groups, controlling the obstructions to constructing consistent interactions, should vanish in the general case, which in turn leads to strong restrictions on the form interaction terms in the Lagrangian description can take.

ZUSAMMENFASSUNG

In dieser Arbeit untersuchen wir welche perturbativen Wechselwirkungen im Lagrangeformalismus von Eichtheorien mit Feldern höheren Spins (HS-Theorien) auftreten können. Der Ausgangspunkt ist die metrikartige Formulierung einer HS-Theorie mit beliebigem ganzzahligen Spin in einer Minkowski-Raumzeit beliebiger Dimension. Unser Ziel ist es, alle widerspruchsfreien, lokalen Deformationen erster Ordnung der freien Theorie zu bestimmen, wobei einzig Poincaré- und Paritätsinvarianz die zu betrachtenden Wechselwirkungsterme einschränken. Zu diesem Zweck verwenden wir BRST-BV kohomologische Methoden, die es durch die Berechnung bestimmter Kohomologiegruppen ermöglichen, Aussagen über die erlaubte Form aller möglichen Wechselwirkungsterme zu tätigen. Das Ergebnis der Arbeit ist eine grobe Beweisskizze, die zeigt, warum entscheidende Kohomologiegruppen verschwinden müssen. Das Verschwinden dieser Gruppen impliziert wiederum starke Einschränkungen der erlaubten Wechselwirkungen.

Acknowledgements

First of all, I would like to express my sincere gratitude and appreciation to my supervisor Prof. Stefan Fredenhagen for his support and guidance, his insight, his advice, his patience, and for giving me the opportunity to be part of the Mathematical Physics Group at the University of Vienna. It was a great pleasure to be part of this group and I want to thank all of its members for our numerous engaging and thought-provoking discussions.

Finally, I want to thank my amazing family, who have encouraged and supported me during my studies and throughout my whole life. None of this work would have been possible without you!

CONTENTS

1 Introduction **1**

2 Mathematical Preliminaries **3**
2.1 Multi-Indices . . . 3
2.2 Graded Algebra . . . 4
2.3 Homological Algebra . . . 7
2.4 Affine Spaces . . . 9
2.5 Derivations . . . 11

3 The BRST Symmetry **13**
3.1 Constrained Hamiltonian Systems . . . 13
3.2 The Longitudinal Complex . . . 18
3.3 The Koszul-Tate Resolution . . . 21
3.4 Extended Phase Space . . . 22
3.5 The BRST Generator . . . 23
3.6 Dynamics of the Ghosts . . . 25
3.7 Gauge Fixing . . . 26

4 Jet Bundles **29**
4.1 The Vertical Bundle and the Transverse Bundle . . . 29
4.2 First-Order Jet Bundles . . . 32
4.3 Prolongation of Morphisms . . . 36
4.4 Higher-order Jet Bundles . . . 37
4.5 The Horizontal Complex . . . 37
4.6 Contact Structure . . . 38

5 The Antifield Formalism **47**
5.1 Lagrangian Field Theories . . . 47
5.2 Local Functionals . . . 48
5.3 Antifields . . . 50
5.4 Ghosts . . . 52
5.5 The Antibracket Structure . . . 53
5.6 The Master Equation . . . 55
5.7 Auxiliary Fields . . . 56

6 Deformations of the Master Equation **57**
6.1 Consistent Couplings as Deformations . . . 58
6.2 Cohomological Reformulation . . . 59

7 Higher-Spin Gauge Theory **61**
7.1 The Fronsdal Formulation . . . 61
7.2 BRST-Formulation . . . 63
7.3 Longitudinal Exterior Cohomology . . . 67
7.4 Invariant Cohomology of δ modulo d . . . 72

8 Conclusion **83**

Bibliography **85**

CHAPTER 1

INTRODUCTION

The antifield formalism, also called Batalin-Vilkovisky (BV) formalism [BV81], [BV83], has established itself as one of the most powerful and elegant methods for quantizing gauge theories. Originally, it was introduced to determine and study the ghost structure of Lagrangian gauge theories, whose constraints do not form a Lie algebra in the Hamiltonian formulation, most prominently gravity and supergravity. In a sense, one arrives at the antifield formalism by generalizing the original BRST formalism [BRS75], [Tyu75] from Yang-Mills theory to arbitrary Lagrangian gauge theories. Over time the formalism has developed from a humble set of working rules, into a complete description of gauge invariance [Fis+89], [Hen90b].

Surprisingly, the problem of perturbatively constructing consistent interactions between fields with gauge freedom, admits a useful reformulation in the antifield formalism [BH93]. In this approach, nontrivial consistent couplings are related to deformations of the solution to the master equation, which can be efficiently analyzed using powerful BRST cohomological deformation techniques.

Compared to earlier approaches, this method offers a lot more generality in the considered perturbations, in that, it does not reject a priori whole classes of possibly relevant perturbation candidates. In fact, even though on physical grounds we shall impose a locality condition, as well as Poincaré and parity invariance, these restrictions will actually add complexity to our discussion, rather than simplify it. For instance, the very general analysis [BBvD84] on spin three self-interactions did not consider cubic vertices containing more than three derivatives. This turned out to be too restrictive, as just such a term, containing five derivatives, was found in [BBC06], using the aforementioned antifield methods.

The problem of introducing consistent local interactions among higher-spin gauge fields is quite old [Fro78], and even after decades of effort it is not generally known under which conditions such interactions are allowed. In general, such interactions are believed to be forbidden, at least in Minkowski spacetime. In curved spacetimes, or spacetimes with non-vanishing cosmological constant, several powerful no-go theorems, restricting possible interactions, fail to hold. The reasons for this are that in curved spacetimes the number of derivatives cannot be used as a proper grading, and a non-vanishing cosmological constant Λ can be used to cancel positive mass dimensions, appearing due to the number of derivatives contained in the vertices.

In the spin three case, in Minkowski spacetime the possible interactions have been strongly constrained recently, among others by [BBC06] and [BBL10]. However, similarly strong restrictions on fields of even higher spin have so far not been established rigorously. The aim of this thesis is to continue in this direction and begin work towards generalizing the results obtained in [Bou+01] and [BBC06] to arbitrary higher spin. A critical step in their analysis is showing that certain cohomology groups, controlling the possible form of interactions, vanish. The computation of these cohomology groups will be our main focus, and at the end of this work, we give an elementary but non-rigorous argument, why they should indeed vanish. We hope with more work this argument can be made rigorous.

In previous approaches, these cohomology groups were shown to vanish by constructing explicit homotopy formulas. These constructions become difficult to generalize to theories of even higher spin, let alone to the case of arbitrary integer spin. Due to the simple structure of the considered cohomologies, we believe a direct computation of the relevant cohomology groups is possible by explicitly constructing the kernels and images of the involved differentials. To make the computations tractable we employ the use of generating functions, as is done for instance in [FKM20], which allows us to argue for all possible integer values of spin simultaneously.

It has been established that fiber bundles, more specifically jet bundles, are the natural framework in which to discuss Lagrangian field theories. Incidentally, the reformulation of quantum field theory in the language of fiber bundles is what led to the realization of the fundamentally geometric nature of the BRST transformation in the first place. Because eventually, we will be restricting our focus to theories in Minkowski spacetime anyways, a description in terms of jet bundles will not be strictly necessary. Nonetheless, even in this case, an understanding of the general framework is beneficial in understanding the fundamental concepts constituting the antifield formalism as well as recognizing issues arising specifically in Minkowski spacetime. Of course, it is indispensable when trying to generalize to more general topologies or geometries. For this reason, a not insignificant portion of this work is devoted to introducing the theory of jet bundles and the necessary mathematical prerequisites.

Moreover, the theory of jet bundles is not the only complication. The multitude of (co)homological techniques, applied all throughout the antifield formalism, requires a significant amount of knowledge of homological algebra. In order for this work to remain somewhat self-contained, we will briefly touch upon the necessary facts in Chapter 2. Because, to those unfamiliar, the complications arising in the infinite dimensional case could cloud the fundamental ideas, we will first introduce the relevant BRST techniques in the context of a theory with a finite number of degrees of freedom in the Hamiltonian formalism in Chapter 3, before moving to the antifield formalism proper in Chapter 5. To simplify presentation we will only ever consider irreducible gauge theories. For the reducible case we refer to the quoted literature in the respective chapters.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

We begin by introducing several important mathematical concepts, necessary to understand the following chapters. To the reader unfamiliar with these topics we suggest consulting the quoted literature. This chapter is not meant to serve as a pedagogical introduction, but rather as a quick refresher and a convenient way to introduce non-standard notation. Throughout this work we shall always consider a D-dimensional spacetime, described by a smooth manifold M of dimension D. We usually denote coordinates on M by $x^0, \dots, x^{D-1}$. Additional structure on M is introduced whenever necessary.

2.1 MULTI-INDICES

Definition 2.1.1 (Multi-Index)**.** A *multi-index* is a D-tupel α of natural numbers. The components of α are denoted by $\alpha(\mu)$ for $0 \leq \mu \leq D-1$. We shall use the letters $\alpha, \beta, \gamma, \dots$ for multi-indices and the letters $\mu, \nu, \rho, \dots$ for ordinary spacetime indices. Addition and subtraction of multi-indices are defined pointwise

$$(\alpha \pm \beta)(\mu) := \alpha(\mu) \pm \alpha(\mu),$$

but notice that subtraction might not result in another multi index. We further define the operations

$$|\alpha| := \sum_{\mu=0}^{D-1} \alpha(\mu), \qquad \alpha! := \prod_{\mu=0}^{D-1} \alpha(\mu)!$$

and the particular multi-indices $1_\nu(\mu) := \delta_{\mu\nu}$. We can use multi-indices to efficiently write commonly occurring derivatives:

$$\partial_\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x^\alpha} := \prod_{\mu=0}^{D-1} \left(\frac{\partial}{\partial x^\mu} \right)^{\alpha(\mu)},$$

where for $\alpha = 0$ this operator shall denote the identity. If we remember that multi-indices are always written using letters from the beginning of the Greek alphabet, we can allow ourselves the following abuse of notation:

$$\partial_{1_\mu} = \frac{\partial}{\partial x^\mu} =: \partial_\mu. \tag{2.1}$$

The following lemma can be proven quite easily using induction.

Lemma 2.1.2. For all sufficiently smooth functions $f, g \in \mathcal{C}^{|\alpha|}(\mathbb{R}^m)$ we use multi-indices to write the usual product rule as

$$\frac{\partial^{|\alpha|}(fg)}{\partial x^\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^{|\beta|} f}{\partial x^\beta} \frac{\partial^{|\gamma|} g}{\partial x^\gamma} \equiv \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial_\beta f \, \partial_\gamma g.$$

2.2 GRADED ALGEBRA

These introductory remarks on graded algebra, or more aptly on *Grassmann-graded differential calculus*, are based on the corresponding section of the book [GMS09]. This book is a great resource on the description of Lagrangian field theories using BRST techniques in general. Unsurprisingly, the exposition of the underlying mathematical structure is ideally suited to our purposes.

Definition 2.2.1 (G-graded Ring)**.** Let G be a group. A *G-graded ring* is a ring R with a direct sum decomposition

$$R = \bigoplus_{g \in G} R_g,$$

such that $R_g R_h \subseteq R_{gh}$. Elements of R that lie inside R_g for some $g \in G$ are called *homogeneous* of *grade* g. For homogeneous elements $r_g \in R_g$ we shall furthermore define the symbol $[r_g] \coloneqq g$.

Definition 2.2.2 (G-graded Algebra)**.** An algebra A over a ring R is a *G-graded algebra* if it is graded as a ring.

Remark. The groups most commonly occurring in typical examples are $\mathbb{Z}$ and $\mathbb{Z}_2$. In fact, in this work we are only interested in the latter and from now on every grading is tacitly assumed to be $\mathbb{Z}_2$. We shall call a $\mathbb{Z}_2$-grading a *Grassmann grading*.

A Grassmann graded algebra A falls into a direct sum $A = A_0 \oplus A_1$. One calls A_0 and A_1 the *even* and *odd* parts of A respectively. A_0 is a subalgebra of A and A_1 is an A_0-module. If A is a graded ring, then $[1] = 0$. We shall also define the *grading automorphism* γ by

$$\gamma(a) \coloneqq a_0 - a_1,$$

where a_0 and a_1 are the homogeneous parts of $a \in A$. Notice that the eigenspaces of γ determine the grading of A and that $\gamma^2 = \mathrm{id}$. We could have started by providing a grading automorphism γ that satisfies $\gamma^2 = \mathrm{id}$ and which defines the grading of A. This is in fact what we shall do when we define graded modules.

Definition 2.2.3 (Graded commutative algebra)**.** A Grassmann graded algebra A is called *graded commutative* if

$$ab = (-1)^{[a][b]} ba,$$

where a and b are homogeneous elements of A.

A real graded commutative ring is said to be of *rank m* if it is a free algebra generated by the unit element 1 and m odd elements.

Definition 2.2.4 (Grassmann Algebra)**.** Let V be a real vector space, and let $\Lambda = \bigwedge V$ be its exterior algebra endowed with the Grassmann grading $\Lambda = \Lambda_0 \oplus \Lambda_1$, where

$$\Lambda_0 = \bigoplus_{k=0}^{\infty} \bigwedge^{2k} V, \qquad \Lambda_1 = \bigoplus_{k=0}^{\infty} \bigwedge^{2k+1} V,$$

under the identification $\bigwedge^0 V \equiv \mathbb{R}$ and $\bigwedge^n V \equiv 0$ for $n > \dim V$. Λ is a real graded commutative ring, called the *Grassmann algebra*. If $\dim V = m < \infty$ then m is the rank of the algebra.

Remark. The Grassmann algebra admits the decomposition

$$\Lambda = \mathbb{R} \oplus \mathcal{R},$$

where $\mathcal{R} = \Lambda_1 \oplus (\Lambda_1)^2$ is the ideal of nilpotents in Λ. The corresponding projections

$$b \colon \Lambda \longrightarrow \mathbb{R},$$
$$s \colon \Lambda \longrightarrow \mathcal{R}$$

are called *body* and *soul* maps respectively.

Definition 2.2.5 (Graded modules)**.** Let A be a graded algebra. A left A-module Q together with a grading automorphism γ such that

(i) $\gamma(aq) = \gamma(a)\gamma(q)$ for all $a \in A$ and $q \in Q$,

(ii) $[aq] \equiv [a] + [q] \pmod 2$
for all homogeneous elements $a \in A$ and $q \in Q$,

is called a left *graded A-module*. Right graded A-modules are defined analogously.

Remark. A graded A-module Q splits into the direct sum

$$Q = Q_0 \oplus Q_1,$$

of the two A_0-modules Q_0 and Q_1 which contain the even and odd elements respectively. Notice that the algebra structure of A plays no role in the definition and only the underlying unital ring structure of A is used. This means one can also talk about graded K-modules, where K is a graded commutative ring. Such a graded K-module can furthermore be provided with a graded K-bimodule structure by letting

$$kq = (-1)^{[k][q]} qk,$$

for all homogeneous elements $k \in K, q \in Q$ and extending by linearity.

Definition 2.2.6 (Graded morphism)**.** Let K be a graded commutative ring, and P and Q be graded K-modules. A group homomorphism $\phi \colon P \to Q$ is called an *even graded morphism* if ϕ preserves the Grassmann parity of all homogeneous elements of P and obeys the relation

$$\phi(kp) = (-1)^{[\phi][k]} k\phi(p),$$

for all $p \in P$ and homogeneous $k \in K$. If it changes the Grassmann parity it is called an *odd graded morphism*.

A group homomorphism $\phi \colon P \to Q$ is called a *graded K-module morphism* if it is represented by a sum of even and odd graded morphisms.

Definition 2.2.7 (K-ring)**.** A graded K-module that is also a graded commutative ring is called *K-ring*.

Let K be a graded commutative ring, A a graded commutative K-ring, and Q and P graded A-modules. The set $\mathrm{Hom}_K(P,Q)$ of graded morphisms is a graded K-module. it can be equipped with two graded A-module structures:

(i) $(a\phi)(p) \coloneqq a\phi(p)$,

(ii) $(\phi \cdot a)(p) \coloneqq \phi(ap)$,

where $a \in A$, $p \in P$ and $\phi \in \mathrm{Hom}_K(P,Q)$. This allows us to define the operator

$$\delta_a\phi \coloneqq a\phi - (-1)^{[a][\phi]}\phi \cdot a,$$

where $a \in A$ is a homogeneous element.

Definition 2.2.8 (Graded differential operator)**.** An element $\Delta \in \mathrm{Hom}_K(P,Q)$ is called a Q-valued *graded differential operator* of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s}\Delta = 0,$$

for any $s+1$ homogeneous elements $a_0, \ldots, a_s \in A$. The set $\mathrm{Diff}_s(P,Q)$ of these operators inherits the graded A-module structures.

The most important special case for us arises when $P = A$ and $s = 1$.

Definition 2.2.9 (Graded Derivation)**.** A first oder Q-valued differential operator $\partial \in \mathrm{Diff}_1(A,Q)$ on A is called a Q-valued *homogeneous derivation* if $\partial(1) = 0$. where $\epsilon = \pm 1$ is called *commutator factor*. A sum of homogeneous derivations with the same ϵ is called *graded derivation*.

Remark. A Q-valued graded derivation ∂ satisfies the Grassmann-graded Leibnitz rule

$$\partial(ab) = \partial(a)b + (-1)^{[A][\partial]}a\partial(b),$$

for every homogeneous element $a \in A$ and every element $b \in A$, and clearly, any first order graded differential operator Δ on A can be written as a sum

$$\Delta(a) = \Delta(1)a + (\Delta(a) - \Delta(1)a)$$

of a zero order graded differential operator $\Delta(1)a$ and a graded derivation $\Delta(a) - \Delta(1)a$.

If ∂ is a graded derivation on A, then so is $a\partial$ for any $a \in A$, hence graded derivations on A constitute a graded A-module.

Definition 2.2.10 (Graded derivation module)**.** The graded A-module of all graded derivations on A is called *graded derivation module* on A and denoted by $\partial(A,Q)$.

Remark. If $Q = A$ then the graded derivation module $\partial A \equiv \partial(A,A)$ is a Lie superalgebra over K with respect to the superbracket

$$[u,v] \coloneqq u \circ v - (-1)^{[u][v]}v \circ u,$$

for $u, v \in \partial A$, and we have the graded A-module decompostion

$$\mathrm{Diff}_1(A) = A \oplus \partial A$$

as can easily be seen from the previous remark.

2.3 HOMOLOGICAL ALGEBRA

For what follows we shall need a basic understanding of (co-)chain complexes and their associated (co-)homology. We begin by defining the standard notion of a chain complex and maps of chain complexes.

Definition 2.3.1 (Chain Complex)**.** A *chain complex* is a sequence of objects with abelian structure (e.g. modules, vector spaces)

$$\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots,$$

with connecting homomorphisms $d_k \colon A_k \to A_{k-1}$ such that $d_{k-1} \circ d_k = 0$ for all $k \in \mathbb{Z}$. Together, this data is denoted $(A_\bullet, d_\bullet)$.

A *cochain complex* is also a sequence of objects with abelian structure

$$\ldots, A^{-2}, A^{-1}, A^0, A^1, A^2, \ldots,$$

with connecting homomorphisms $d^k \colon A^k \to A^{k+1}$ such that $d^{k+1} \circ d^k = 0$ for all $k \in \mathbb{Z}$. Together, this data is denoted $(A^\bullet, d^\bullet)$.

Remark. In most cases relevant for us we will be considering a sequence of vector spaces. Additionally, the sequences will almost always be one sided, i.e. $A^k = \emptyset$ for $k < 0$. We will mostly be considering cochain complexes, so the following definitions and results are formulated in terms of those, however they apply, mutatis mutandis, equally well to chain complexes.

Definition 2.3.2 (Map of Complexes)**.** A *map of complexes* $\phi^\bullet \colon (A^\bullet, d^\bullet) \to (B^\bullet, e^\bullet)$ is a collection of maps $\phi^k \colon A^k \to B^k$ such that $e^k \circ \phi^k = \phi^{k+1} \circ d^k$. To ease notation we shall always denote all d^k and e^k simply by d.

Definition 2.3.3 (Closed and Exact)**.** An element $\omega \in A^k$ is called *closed* if $d\omega = 0$. It is called *exact* if there exists $\eta \in A^{k-1}$ such that $\omega = d\eta$. Since $d \circ d = 0$, all exact elements are also closed.

The most important and well-known example is probably the *de Rham complex*. Given a smooth manifold M, we denote the space of p-forms on M by $\Omega^p(M)$. The exterior derivative d serves as the differential for the cochain complex

$$0 \to \mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots.$$

Generally, it is natural to ask if there exist closed elements which are not exact, and subsequently, how many different such elements exist. These questions lead one to define the following structures:

Definition 2.3.4 (Homology and Cohomology)**.** The k^{th} *homology group* of a chain complex $(A_\bullet, d_\bullet)$ is defined to be the quotient

$$H_k(A_\bullet) := \frac{\ker d_k}{\operatorname{im} d_{k+1}}.$$

Analogously, the k^{th} *cohomology group* of a cochain complex $(A^\bullet, d^\bullet)$ is defined to be the quotient

$$H^k(A^\bullet) := \frac{\ker d^k}{\operatorname{im} d^{k-1}}.$$

We notice that the k^{th} (co)homology group describes the equivalence classes of d^k-closed elements, where we consider two closed elements to be equivalent if their difference is exact.

Lemma 2.3.5. Consider cochains $A^\bullet, B^\bullet$ and a map of complexes $\phi^\bullet \colon A^\bullet \to B^\bullet$. Then $\phi^\bullet$ induces maps

$$H^k(\phi^\bullet) \colon H^k(A^\bullet) \longrightarrow H^k(B^\bullet).$$

Proof. The map

$$\begin{aligned} H^k(\phi^\bullet) \colon H^k(A^\bullet) &\longrightarrow H^k(B^\bullet) \\ [w] &\longmapsto [\phi(\omega)] \end{aligned}$$

is well-defined, because

$$[\phi(\omega + d\eta)] = [\phi(\omega) + d\phi(\eta)] = [\phi(\omega)]. \qquad \square$$

We will now describe under which circumstances two maps of complexes induce the same map between cohomology groups.

Definition 2.3.6 (Homotopy)**.** Let X, Y be topological spaces and $f, g \colon X \to Y$ two continuous maps. A *homotopy* between f and g is a continuous map

$$H \colon X \times [0,1] \longrightarrow Y,$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Definition 2.3.7 (Algebraic Homotopy)**.** Two maps $\phi^\bullet, \psi^\bullet \colon A^\bullet \to B^\bullet$ of complexes are called *homotopic* through the *homotopy* $h^\bullet$ if there exists maps $h^k \colon A^k \to B^{k-1}$ such that

$$\phi^k - \psi^k = d \circ h^k + h^{k+1} \circ d.$$

Proposition 2.3.8. If two maps $\phi^\bullet, \psi^\bullet$ are homotopic through some homotopy $h^\bullet$, then

$$H^k(\phi^\bullet) = H^k(\psi^\bullet).$$

Proof. By linearity we have

$$\begin{aligned} H^k(\phi^\bullet) - H^k(\psi^\bullet) &= H^k(\phi^\bullet - \psi^\bullet) \\ &= H^k(d \circ h^\bullet + d \circ h^\bullet) \\ &= H^k(d \circ h^\bullet) + H^k(d \circ h^\bullet). \end{aligned}$$

It is easy to see that both $H^k(d \circ h^\bullet)$ and $H^k(h^\bullet \circ d)$ vanish. $\square$

Definition 2.3.9 (Resolution)**.** Let A be an algebra. A *homological resolution* of A is a graded differential algebra $\widetilde{A}$ with differential δ of degree minus one, such that

$$H_k(\delta) \cong \begin{cases} A & k = 0, \\ 0 & k \geq 1. \end{cases}$$

We see that a resolution represents A as the quotient algebra $H_\bullet(\delta) \cong H_0(\delta)$. The grading on $\widetilde{A}$ is called *resolution degree*. A *cohomological resolution* is defined analogously.

2.4 AFFINE SPACES

This very brief summary of affine spaces and their relevant properties for our discussion of affine bundles in Section 4.2 is based on the excellent article [MMS02] by Martinez et al.

Definition 2.4.1 (Affine Space)**.** Let V be a vector space, A a set and $\alpha\colon A \times V \to A$. Then the triple (A, V, α) is called an *affine space* if:

(i) $\alpha(x, 0) = x$, for each $x \in A$

(ii) $\alpha(\alpha(x, v), w) = \alpha(x, v + w)$ for each $x \in A$ and each $v, w \in V$.

(iii) for $x, y \in A$ there is a unique $v \in V$ satisfying $\alpha(x, v) = y$.

Definition 2.4.2 (Affine Morphisms)**.** Let A, B be affine spaces modeled on V, W by the maps α, β respectively. A function $T\colon A \to B$ is called *affine morphism* if there is a linear map $\overline{T}\colon V \to W$ such that

$$T(\alpha(x, v)) = \beta(T(x), \overline{T}(v)),$$

for all $x \in A$ and $v \in V$.

Remark. The map $\overline{T}$ is called the *linear part* of T. An affine morphism which is invertible is called an *affine isomorphism* because its inverse is also an affine morphism: in fact T is an affine isomorphism if, and only if, its linear part $\overline{T}$ is a vector space isomorphism, and then $\overline{T^{-1}} = \overline{T}^{-1}$. An affine morphism from A to itself whose linear part is id_V is called a *translation*.

Definition 2.4.3 (Extended Dual of Affine Spaces)**.** Let A be an affine space modeled on a vector space V. We define the *extended dual of* A to be the vector space of all affine maps from A to the real line. We shall mostly denote it as $A^\dagger := \mathrm{Aff}(A, \mathbb{R})$.

We know that for a vector space V, the bidual space $V^{**} := (V^*)^*$ is naturally isomorphic to V. The *bidual* $A^{**} := (A^\dagger)^*$ of A induces a copy of A in the following sense.

Lemma 2.4.4. The inclusion map $i\colon A \to A^{**}$ given by $i_a(\phi) := \phi(a)$ is an injective affine map, with associated vector map $\vec{i}\colon V \to A^{**}$ given by $\vec{i}_v(\phi) := \vec{\phi}(v)$.

Proof. For all $a \in A, v \in V$ and $\phi \in A^\dagger$ we have

$$i_{a+v}(\phi) = \phi(a + v) = \phi(a) + \vec{\phi}(v) = i_a(\phi) + \vec{i}(v),$$

so i is an affine map with associated linear map $\vec{i}$. The vector map $\vec{i}$ is clearly injective since for $v \in \ker \vec{i}$ we have

$$\vec{\phi}(v) = \vec{i}_v(\vec{\phi}) = 0$$

for all $\vec{\phi} \in V^*$ which implies $v = 0$. This already shows that i is injective. □

To better understand the structure of A^{**} we will show that it fits into an exact sequence of the form

$$0 \longrightarrow V \longrightarrow A^{**} \longrightarrow \mathbb{R} \longrightarrow 0.$$

Lemma 2.4.5. Let $f\colon \mathbb{R} \to A^\dagger$ be the map that associates to $\lambda \in \mathbb{R}$ the constant function λ on A. Let $g\colon A^\dagger \to V^*$ be the map that associates to every affine function on A the corresponding linear function on V. The sequence of vector spaces

$$0 \longrightarrow \mathbb{R} \xrightarrow{f} A^\dagger \xrightarrow{g} V^* \longrightarrow 0$$

is exact.

Proof. Clearly f is injective, g is surjective and $g \circ f = 0$, so that $\operatorname{im} f \subseteq \ker g$. If $\phi \in \ker g$ this means that the linear part of ϕ vanishes and so for every $a, b \in A$

$$\phi(b) = \phi(a + v) = \phi(a) + \vec{\phi}(v) = \phi(a),$$

where $v \coloneqq b - a$. We see that ϕ is constant and hence lies in the image of f. □

What is left to do is take the dual of this sequence. Since

$$\langle g(\phi), v\rangle = \langle \vec{\phi}, v\rangle = \langle \phi, \vec{i}(v)\rangle,$$

for every $v \in V$, we identify the dual map of g with $\vec{i}$. We denote by $j\colon A^{**} \to \mathbb{R}$ the dual of f. Since for every $\lambda \in \mathbb{R}$ and $z = \alpha i(a) + \vec{i}(v) \in A^{**}$ we have

$$\begin{aligned} j(z)\lambda &= \langle z, f(\lambda)\rangle \\ &= \langle \alpha i(a) + \vec{i}(v), f(\lambda)\rangle \\ &= \alpha\langle i(a), f(\lambda)\rangle + \langle \vec{i}(v), f(\lambda)\rangle \\ &= \alpha\lambda, \end{aligned}$$

so that j is characterized by $j(\alpha i(a) + \vec{i}(v)) = \alpha$. We have thus shown the following corollary.

Corollary 2.4.6. Let A be a finite dimensional affine space modeled on V. The sequence

$$0 \longrightarrow V \xrightarrow{i} A^{**} \xrightarrow{j} \mathbb{R} \longrightarrow 0$$

is exact.

Remark. We get the convenient identifications

$$\begin{aligned} \vec{i}(V) &= j^{-1}(\{0\}), \\ i(A) &= j^{-1}(\{1\}). \end{aligned}$$

Note that we get a characterization of affine spaces in terms of exact sequences for free.

Corollary 2.4.7. Given an exact sequence

$$0 \longrightarrow V \xrightarrow{\alpha} W \xrightarrow{j} \mathbb{R} \longrightarrow 0$$

of vector spaces V and W, we define $A \coloneqq j^{-1}(\{1\})$. Then A is an affine space modeled on V, and W is canonically isomorphic to A^{**}. The isomorphism is the dual to the map $\Phi\colon W^* \to A^\dagger$ given by $\Phi_\phi(a) \coloneqq \phi(i(a))$, where $i\colon A \to W$ is the canonical inclusion.

2.5 DERIVATIONS

In the context of a fiber bundle $\pi\colon E \to M$, we are interested in derivations mapping differential forms on M to differential forms on E.

Definition 2.5.1 (Derivation along π)**.** A *derivation along* π of degree r is an $\mathbb{R}$-linear map

$$\Delta\colon \bigwedge M \longrightarrow \bigwedge E,$$

satisfying

(i) if $\vartheta \in \bigwedge^s M$ then $\Delta\vartheta \in \bigwedge^{r+s} E$,

(ii) if $\vartheta_1 \in \bigwedge^{s_1} M$, $\vartheta_2 \in \bigwedge^{s_2} M$ then

$$\Delta(\vartheta_1 \wedge \vartheta_2) = \Delta\vartheta_1 \wedge \pi^\star\vartheta_2 + (-1)^{rs_1}\pi^\star\vartheta_1 \wedge \Delta\vartheta_2.$$

We shall distinguish between two particular types of derivations.

Definition 2.5.2 (Derivation of Type $\imath_\star$)**.** A derivation Δ along π is *of type* $\imath_\star$ if $\Delta f = 0$ for all $f \in \mathcal{C}^\infty(M) \simeq \bigwedge^0 M$.

Definition 2.5.3 (Derivation of Type $d_\star$)**.** A derivation Δ along π of degree r is *of type* $d_\star$ if

$$\Delta \circ d = (-1)^r\, d \circ \Delta,$$

where on the left side d is the exterior derivative on M, while on the right side d is the exterior derivative on E.

Derivations of type $\imath_\star$ are modeled on interior contractions, while those of type $d_\star$ are modeled on the Lie derivative.

CHAPTER 3

THE BRST SYMMETRY

In this chapter we will make first contact with the so-called BRST[1] symmetry and we begin by discussing the BRST construction in the simplest possible setting, namely a classical irreducible gauge theory, with a finite number of degrees of freedom, in the Hamiltonian description. Eventually we will want to extend the concepts introduced in this chapter to the Lagrangian description of field theories, but for the moment we wish to avoid the mathematical subtleties lurking in those theories until we have become more accustomed to the underlying ideas of the BRST construction.

We will not discuss reducible gauge theories in this work at all. Mainly because the theory which we will consider in Chapter 7 is irreducible, and discussing the reducible case does not add to the clarity of the presentation. Moreover, including the reducible case would not require any new ideas, it would just add to the complexity of the explicit constructions.

This chapter is basically a summary of the relevant results described in [HT92], streamlined for our purposes in the following chapters.

3.1 CONSTRAINED HAMILTONIAN SYSTEMS

Even though we shall quickly switch to the Hamiltonian formulation, it serves to start out in a Lagrangian formulation of the theory described by an action S. So, consider a system whose classical motions are those that make the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt \tag{3.1}$$

stationary under variations of the Lagrangian variables $q^1, \ldots q^n$, which we require to vanish at the boundaries t_1 and t_2. The conditions for the action to be stationary are the Euler-Lagrange equations

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \equiv \ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} + \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} - \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n. \tag{3.2}$$

[1] named after Becchi, Rouet, Stora [BRS75], and Tyutin [Tyu75]

Equation (3.2), considered as a vector equation, shows that at any given time, the accelerations $\ddot{q}^i$ are uniquely determined by the positions and velocities, if and only if the determinant

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\right) \tag{3.3}$$

is not equal to zero. Should this determinant vanish, then the accelerations will not be uniquely determined by the positions and the velocities, and the solutions of the equations of motion could contain arbitrary functions of time. Moving to the Hamiltonian formalism, the canonical momenta are defined as

$$p_i := \frac{\partial L}{\partial\dot{q}^i} \tag{3.4}$$

and the vanishing of the determinant (3.3) translates to the non-invertibility of the velocities as functions of the coordinates and momenta. This means that the momenta (3.4) are not all independent, but rather they obey some relations

$$\phi_m(q,p) = 0, \quad m = 1,\ldots,M, \tag{3.5}$$

that follow directly from their explicit form, as given by (3.4). These conditions are called primary constraints to emphasize that the equations of motion are not used to obtain them.

We have to assume that the matrix $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$ has constant rank throughout $(q,\dot{q})$ space, and that the primary constraints (3.5) define an embedded submanifold Σ in phase space. The submanifold Σ is called *constraint surface*. Now, despite the fact that the velocities can not be written as functions of the coordinates, they enter the expression

$$\widetilde{H}(q,\dot{q}) := \dot{q}^i p_i(q,\dot{q}) - L(q,\dot{q}) \tag{3.6}$$

only through the combination $p(q,\dot{q})$, as is easily shown by the computation of

$$\begin{aligned}\delta\widetilde{H} &= \delta\dot{q}^i p_i + \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial\dot{q}^i}\delta\dot{q}^i \\ &= \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i,\end{aligned}$$

where δp_i is regarded as a linear combination of the δq and $\delta\dot{q}$, and not an independent variation. Hence, $\delta\dot{q}$ only appears in the combination δp and so there exists a function $H(q,p)$ such that

$$H(q,p(q,\dot{q})) = \widetilde{H}(q,\dot{q}). \tag{3.7}$$

Notice however that the function H is not unique, since the addition of primary constraints

$$H \longrightarrow H + u^m\phi_m, \tag{3.8}$$

where the u^m are arbitrary functions of q and p, does not affect condition (3.7). In other words, the Hamiltonian H is uniquely defined only on the previously defined constraint surface Σ. From now on H shall denote the phase space function obtained by the Legendre transformation of L, and we will call H the *canonical Hamiltonian*.

Now, the derivation of the equivalent equations of motion in their Hamiltonian form is a bit technical, so we will not go into it here, but the fact that they have to be invariant under (3.8) makes it plausible that they should take the form

$$\begin{aligned}\dot{q}^i &= \frac{\partial H}{\partial p_i} + u^m \frac{\partial \phi_m}{\partial p_i},\\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} - u^m \frac{\partial \phi_m}{\partial q^i},\\ 0 &= \phi_m(q,p).\end{aligned}$$

The functions u^m here are basically fixed by the particular choice of Hamiltonian. The equations of motion can equally be written as the variational principle

$$\delta \int_{t_1}^{t_2} \left(\dot{q}^i p_i - H - u^m \phi_m \right) = 0 \tag{3.9}$$

for arbitrary variations $\delta q, \delta p, \delta u$ subject only to the restriction $\delta q(t_1) = \delta q(t_2) = 0$. Notice that the variables u^m now appear as Lagrange multipliers enforcing the primary constraints. Introducing the Poisson bracket

$$[F,G] := \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}, \tag{3.10}$$

where F and G are arbitrary functions of the canonical variables, allows us to write the equations of motion as

$$\dot{F} = [F,H] + u^m [F,\phi_m]. \tag{3.11}$$

Having established time evolution, a basic consistency requirement of the theory is that the primary constraints be preserved in time, which gives rise to the consistency conditions

$$0 = \dot{\phi}_m = [\phi_m, H] + u^{m'} [\phi_m, \phi_{m'}]. \tag{3.12}$$

This might either impose a restriction on the functions u^m or give a relation only involving the p and q. If this relation is not yet included in the primary constraints it is called a *secondary constraint* and the previous consistency condition is also applied to it. This procedure is repeated until there are no more new constraints produced. Because the distinction between primary and secondary constraint is not important for our considerations we shall simply group them together as the collection of constraints

$$\phi_j = 0, \quad j = 1, \ldots, J. \tag{3.13}$$

We make the same regularity assumptions on the full set of constraints ϕ_j as we did on the primary constraints. Among others, a very important assumption is that the matrix $[\phi_i, \phi_j]$ has constant rank everywhere on Σ. Another assumption we make on the constraints is independence. Should the Equations (3.13) be independent, we say that the theory under consideration is an *irreducible* gauge theory. If they are not independent, the theory is called *reducible*. As we mentioned in the introduction to this chapter, we shall only consider irreducible theories, so we assume (3.13) to be independent. For more information please refer to [HT92].

At this point we introduce the notation $\mathcal{C}^\infty(P)$ for the set of smooth functions on phase space and the symbol $\approx$ to denote equality on the constraint surface. Thus, by writing

$$\phi_j \approx 0, \tag{3.14}$$

we emphasize that the quantity ϕ_j is numerically restricted to be zero on Σ, but does not necessarily vanish identically throughout all of phase space. A direct consequence of the usual regularity conditions is that a function $F \in \mathcal{C}^\infty(P)$, that vanishes everywhere on the constraint surface can be written as a combination of the constraints ϕ_j, i.e.

$$F \approx 0 \quad \Leftrightarrow \quad F = c^j(q,p)\phi_j. \tag{3.15}$$

Having found a complete set of constraints we shall now consider the corresponding restrictions on the Lagrange multipliers u^m

$$[\phi_j, H] + u^m[\phi_j, \phi_m] \approx 0, \quad j = 1, \ldots, J, \tag{3.16}$$

where m is summed from 1 to M. Interpreting Equation (3.16) as a set of J non-homogeneous linear equations in the $M \leq J$ unknowns u^m, with coefficients that are functions of q and p, we see that these equations should possess solutions. If they did not, the system described by the Lagrangian L would be inconsistent. The general solution of (3.16) can be written as

$$u^m = U^m + V^m, \tag{3.17}$$

where U^m is a particular solution and V^m is the most general solution of the homogeneous system

$$V^m[\phi_j, \phi_m] \approx 0. \tag{3.18}$$

The most general solution of Equation (3.18) has to be a linear combination of linearly independent solutions V_a^m, $a = 1, \ldots, A$, where the number A of linearly independent solutions is constant on Σ. The fact that the number A is constant on Σ follows from the assumption that the rank of the matrix $[\phi_j, \phi_m]$ is constant on Σ. Therefore, in terms of arbitrary linear coefficients v^a and the non-unique particular solution U^m, the general solution to (3.16) takes the form

$$u^m \approx U^m + v^a V_a^m. \tag{3.19}$$

By writing the solution in this way we have separated the part of u^m that is determined by the consistency conditions, from the part that remains arbitrary. In the spirit of this decomposition we can define

$$\begin{aligned} H' &:= H + U^m\phi_m, \\ \phi_a &:= V_a^m\phi_m, \\ H_\mathrm{T} &:= H' + v^a\phi_a, \end{aligned}$$

which allows us to rewrite the expression for time evolution (3.11) as

$$\dot{F} \approx [F, H' + v^a\phi_a] = [F, H_\mathrm{T}]. \tag{3.20}$$

The function H_T is called *total Hamiltonian*. While the distinction between primary and secondary constraints does not concern us, a different classifications scheme plays a

crucial role. A function $F \in \mathcal{C}^\infty(P)$ is called *first-class* if its Poisson bracket with every constraint vanishes weakly,

$$[F, \phi_j] \approx 0, \quad j = 1, \ldots, J. \tag{3.21}$$

Otherwise it is called *second-class*. Here we will mention without proof, that every constrained Hamiltonian system can be extended, in such a way as to make every constraint first-class, which is why we will focus on this case going forward.

The presence of arbitrary functions v^a in the total Hamiltonian tells us that not all p and q are observable. While a physical state is uniquely specified once a set of p and q is given, the converse is not true. To see this, consider the difference in time evolution of a function F under two distinct choices v^a and $\tilde{v}^a$:

$$\begin{aligned}\Delta F &:= F(t+\delta t)|_{v^a} - F(t+\delta t)|_{\tilde{v}^a} \\ &\approx (v^a - \tilde{v}^a)[F, \phi_a]\,\delta t + \mathcal{O}(\delta t^2) \\ &\equiv [F, \phi_a]\,\Delta v^a \delta t + \mathcal{O}(\delta t^2),\end{aligned}$$

which, when we note that $F(t+\delta t) = F(t) + \mathcal{O}(\delta t)$, shows that an infinitesimal transformation of the form

$$\delta F := [F, \phi_a]\,\delta v^a, \tag{3.22}$$

cannot alter the physical state of the system. Therefore, we say that the first-class primary constraints generate gauge transformations. In the reducible case, i.e. where some of the constraints $\phi_a \approx 0$ are redundant, some of these transformations can reduce to $\delta F \approx 0$. In a similar manner it can be shown that the Poisson brackets $[\phi_a, \phi_{a'}]$ and $[\phi_a, H']$ also generate gauge transformations. This indicates that there might also be some secondary first-class constraints that act as generators of gauge transformations. While it is not possible to prove from these considerations that every secondary first class constraint is such a generator, because in general it is not true, we nevertheless postulate that all first-class constraints generate gauge transformations. This postulate is called the *Dirac conjecture*.

Notice that the arguments leading to the identification of ϕ_a as generators that do not change the physical state at a given time assume that the time t, the integration variable in the Lagrangian, is observable, which is information brought in from the outside. A different point of view could be that some of the gauge arbitrariness indicates that the time itself is not observable, which is taken in so-called generally covariant theories.

As we have already mentioned, we shall not consider the distinction between primary and secondary constraints, and in general, we can assume all constraints to be first-class. However, to remain consistent with the notation used in [HT92], we introduce the new notation G_a for the first-class constraints (i.e. all of the constraints that we are considering). Every physically permissible motion has to allow for arbitrary gauge transformations to be performed while the system is evolving in time. We should therefore add to the total Hamiltonian H_T the remaining first-class secondary constraints multiplied by additional arbitrary functions

$$H_\mathrm{E} := H' + u^a G_a. \tag{3.23}$$

The resulting first-class function H_E is called *extended Hamiltonian*. For gauge-invariant dynamical variables (i.e. variables such that their Poisson bracket with the gauge generators G_a vanish weakly), the evolution predicted by H', H_T and H_E is of course the

same. For all other variables we should use H_{E} to account for all of the gauge freedom. We emphasize that the need for the extended Hamiltonian does not follow from the Lagrangian theory. It is the total Hamiltonian H_{T} that generates the original Lagrangian equations of motion. The introduction of H_{E} is a new feature of the Hamiltonian scheme, which extends the Lagrangian formalism by making manifest all of the gauge freedom. The equations of motion for the extended formalism can equally well be understood to be derived from the *extended action principle*

$$S_{\mathrm{E}} = \int \left(p_i \dot{q}^i - H' - u^a G_a \right) dt, \tag{3.24}$$

where the sum runs over all constraints, which gives

$$\dot{F} \approx [F, H_{\mathrm{E}}], \qquad \text{and} \qquad G_a \approx 0. \tag{3.25}$$

Because the dynamical system is constrained to be on the constraint surface Σ, two phase space functions cannot be distinguished if they coincide on Σ. In other words, the relevant functions to consider are not all the smooth phase space functions $\mathcal{C}^\infty(P)$ but, rather, the space of smooth functions on Σ, which can algebraically be described as follows. The vector space $\mathcal{C}^\infty(P)$ is an algebra where multiplication is just ordinary pointwise multiplication of functions. Let $\mathcal{N} \subseteq \mathcal{C}^\infty(P)$ be the ideal of smooth functions that vanish on $\Sigma \subseteq P$, i.e. the kernel of the restriction map to Σ. We immediately recognize the algebra isomorphism

$$\mathcal{C}^\infty(\Sigma) \cong {\mathcal{C}^\infty(P)}\big/{\mathcal{N}}, \tag{3.26}$$

which suggests defining classical *observables* as equivalence classes of gauge invariant functions $F \in \mathcal{C}^\infty(P)$ that coincide on Σ. There are two steps to be considered in this definition. First, we need to determine the equivalence classes of functions that coincide on Σ. Secondly, we have to check which of those functions are gauge invariant. The isomorphism (3.26) already suggests an algebraic approach, which we shall pursue starting with the second point.

3.2 THE LONGITUDINAL COMPLEX

Given the algebra $\mathcal{C}^\infty(\Sigma)$, we begin by establishing a suitable description of all gauge invariant functions contained in $\mathcal{C}^\infty(\Sigma)$. Notice, that the Poisson bracket provides us with a symplectic form σ on P. In canonical coordinates $(x^\mu) = (q^i, p_j)$, we can define the components of a contravariant, anti-symmetric tensor

$$\sigma^{\mu\nu} = [x^\mu, x^\nu] = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{3.27}$$

We can now use the tensor $\sigma^{\mu\nu}$ and the constraints G_a to define vector fields X_a on P, by specifying their components with respect to the coordinate basis vectors corresponding to the coordinate functions x^μ:

$$X_a^\mu := \sigma^{\mu\nu} \partial_\nu G_a. \tag{3.28}$$

Accordingly, the vector fields X_a act on functions $F \in \mathcal{C}^\infty(P)$, at this point still defined on all of phase space, via

$$X_a F = X_a^\mu \partial_\mu F = [F, G_a], \tag{3.29}$$

i.e. they generate the infinitesimal gauge transformations. While the vector fields X_a are defined everywhere on P, they are linearly independent (in general) only on Σ.

Definition 3.2.1 (Gauge Distribution)**.** Let $G \subseteq T\Sigma$ be the M-dimensional distribution spanned by the vector fields X_a. We call G the *gauge distribution* and the vector fields X_a naturally form a basis of G.

It is straightforward to check that on Σ the Lie bracket $[X_a, X_b]$ can again be written as a linear combination of the X_c, i.e. G is involutive. This means that G satisfies the Frobenius integrability condition, and hence, there is an M-dimensional foliation of Σ to which G is tangent.

Definition 3.2.2. The leaves of this foliation are called *gauge orbits*.

From the explicit form (3.27), we see that $\det \sigma^{\mu\nu} \neq 0$. Therefore, we can invert $\sigma^{\mu\nu}$, which yields the components of the 2-form σ, satisfying

$$\sigma^{\mu\rho}\sigma_{\rho\nu} = \delta^{\mu}_{\nu}. \tag{3.30}$$

It turns out that the 2-form σ satisfies all properties required to be a symplectic form. Let $i\colon \Sigma \to P$ denote the previously discussed embedding of the constraint surface into phase space, we call $i^\star\sigma$ the *induced 2-form* on Σ. While $\det \sigma \neq 0$, the same is not true for $i^\star\sigma$ and because the vectors X_a are null eigenvectors of the induced 2-form, we also call the M-dimensional surfaces generated by them *null surfaces*.

With a bit more work, it can be established that the gauge orbits coincide with the null surfaces of the induced 2-form. This result relies strongly on the Dirac conjecture. If not all first-class constraints generate gauge transformations, then the gauge orbits would be strictly smaller than the null surfaces, and there would be null directions not associated to any gauge transformation.

Definition 3.2.3 (Longitudinal vector field)**.** Sections of the gauge distribution G are called *longitudinal vector fields*. We let $\Gamma(G)$ denote the space of longitudinal vector fields. In particular, the vector fields X_a are longitudinal.

Definition 3.2.4 (Longitudinal p-forms)**.** A multilinear, anti-symmetric map

$$\alpha\colon \Gamma(G)^p \longrightarrow \mathcal{C}^\infty(\Sigma) \tag{3.31}$$

is called *longitudinal p-form*.

Denote the ordinary tensor product of multilinear maps by $\otimes$. In analogy to ordinary differential forms, if α is a p-form and β is a q-form, we define the product $\alpha\beta$ as

$$\alpha\beta := (-1)^{pq}\frac{(p+q)!}{p!\,q!}(\alpha\otimes\beta)^{\mathrm{alt}}, \tag{3.32}$$

which is just the ordinary wedge product of differential forms with an additional sign factor $\alpha\beta = (-1)^{pq}\alpha\wedge\beta$. For example, if α and β are 1-forms we have

$$(\alpha\beta)(X,Y) = -\alpha(X)\beta(Y) + \alpha(Y)\beta(X). \tag{3.33}$$

Because G is involutive we can introduce an exterior derivative d, called *longitudinal derivative*, acting on longitudinal forms, characterized by the following defining properties:

(i) $(dF)(X) = XF$, for all $F \in \mathcal{C}^\infty$ and longitudinal vector fields X,

(ii) $d^2 = 0$ and

(iii) $d(\alpha\beta) = \alpha d\beta + (-1)^q (d\alpha)\beta$, for all longitudinal p-forms α and q-forms β.

Explicitly we can write d using the Eilenberg-Cartan formula, where it becomes clear that G being involutive is crucial, because otherwise the Lie bracket of two longitudinal vector fields would in general not be longitudinal again. Of special interest to us are the closed longitudinal forms

$$d\alpha = 0. \tag{3.34}$$

Since $d^2 = 0$, all exact forms

$$\alpha = d\beta \tag{3.35}$$

are closed. The longitudinal cohomology of degree p is defined as the quotient space of the closed p-forms modulo the exact ones. The longitudinal cohomology spaces are usually denoted as

$$H^p(d) = \left(\frac{\ker d}{\operatorname{im} d} \right)^p. \tag{3.36}$$

Notice that $H^0(d)$ are just the gauge invariant functions on Σ. The higher cohomology groups depend on specific properties of the foliation. Their physical role is not well understood. Since in the irreducible case, the vector fields X_a form a basis of G, we can construct everywhere on Σ a basis η^a of longitudinal 1-forms, by the condition

$$\eta^a(X_b) = \delta^a_b, \tag{3.37}$$

for $a, b = 1, \ldots, M$. This enables us to view general longitudinal p-forms α as polynomials in the η^a

$$\alpha = \frac{1}{p!} \alpha_{a_1 \ldots a_p} \eta^{a_1} \cdots \eta^{a_p}, \tag{3.38}$$

with coefficients $\alpha_{a_1 \ldots a_p} \in \mathcal{C}^\infty(\Sigma)$. In this representation d is characterized by the rules

$$dF = (X_a F)\eta^a, \tag{3.39}$$

$$d\eta^a = \frac{1}{2} \eta^b \eta^c C_{cb}{}^a, \tag{3.40}$$

as well as the Leibniz rule, where $[X_a, X_b] = C_{ab}{}^c X_c$ on Σ. While, as we emphasize here again, the longitudinal forms are only defined on the constraint surface Σ, it is useful to give their components as functions on P rather than on Σ, because in practice it is hard to display a coordinate system on Σ. Using the algebra isomorphism

$$\mathcal{C}^\infty(\Sigma) \cong \mathcal{C}^\infty(P) \big/ \mathcal{N} \tag{3.41}$$

we can regard the longitudinal forms as polynomials in η^a

$$\alpha = \frac{1}{p!} \alpha_{a_1 \ldots a_p} \eta^{a_1} \cdots \eta^{a_p}, \tag{3.42}$$

with coefficients $\alpha_{a_1 \ldots a_p} \in \mathcal{C}^\infty(P)$, under the identification

$$\alpha \sim \alpha' \quad \Leftrightarrow \quad \alpha_{a_1 \ldots a_p} \approx \alpha'_{a_1 \ldots a_p}. \tag{3.43}$$

Because the gauge transformations and the structure functions $C_{ab}{}^{c}$ are defined everywhere on P and not just Σ, we can define the exterior derivative in the same way as before, however, one finds that, in general, d^2 is zero only on the constraint surface

$$d^2 \approx 0. \tag{3.44}$$

This is because the vector fields X_a (in general) do not obey the Frobenius integrability condition off Σ. In terms of the description on P, the longitudinal cohomology is obtained by identifying two weakly closed polynomials that differ by a weakly exact one.

3.3 THE KOSZUL-TATE RESOLUTION

Having introduced longitudinal forms, which, in some sense to be specified more later, describe the gauge invariant functions on $\mathcal{C}^\infty(\Sigma)$, our next task is to construct a resolution, cf. Definition 2.3.9, that describes the restriction to the constraint surface Σ, defined by the conditions

$$G_a = 0, \tag{3.45}$$

through its homology. That is, we must construct a differential δ on $\mathcal{C}^\infty(P)$, or possibly some larger space containing $\mathcal{C}^\infty(P)$, such that

$$H_0(\delta) \cong \mathcal{C}^\infty(\Sigma) \tag{3.46}$$

and $H_k(\delta) \cong 0$ for $k \neq 0$. If we wish to fulfill

$$H_0(\delta) \equiv \frac{(\ker \delta)_0}{(\operatorname{im} \delta)_0} \cong \mathcal{C}^\infty(\Sigma) \cong {\mathcal{C}^\infty(P)}\big/{\mathcal{N}}, \tag{3.47}$$

it suggests itself to define δ so that

$$(\ker \delta)_0 = \mathcal{C}^\infty(P) \tag{3.48}$$

and

$$(\operatorname{im} \delta)_0 = \mathcal{N}. \tag{3.49}$$

Achieving (3.48) is easy enough, simply set δ to be zero on $\mathcal{C}^\infty(P)$. To achieve (3.49) we recall that, due to the regularity conditions on the constraints, an arbitrary element in $\mathcal{N}$ can be written as a combination of the constraints with coefficients that are phase space functions,

$$F \approx 0 \quad \Leftrightarrow \quad F = F^a G_a, \tag{3.50}$$

where $F^a \in \mathcal{C}^\infty(P)$. The idea now is to introduce additional Grassmann odd objects $\mathcal{P}_a$, one for each G_a, and to set

$$\delta \mathcal{P}_a := -G_a, \tag{3.51}$$

which gives

$$F \approx 0 \quad \Leftrightarrow \quad F = \delta(-F^a \mathcal{P}_a) \in (\operatorname{im} \delta)_0. \tag{3.52}$$

The inclusion of the minus sign is for later convenience. Hence, by defining δ on $\mathbb{C}[\mathcal{P}_a] \otimes \mathcal{C}^\infty(P)$ in this way and extending it to the whole space by requiring it to be an odd right derivation, we should achieve a resolution of $\mathcal{C}^\infty(\Sigma)$, which is indeed the case as the following theorem states.

Theorem 3.3.1. The differential algebra $(\mathbb{C}[\mathcal{P}_a] \otimes \mathcal{C}^\infty(P), \delta)$ provides a resolution of $\mathcal{C}^\infty(\Sigma)$, i.e.

$$H_k(\delta) \cong \begin{cases} \mathcal{C}^\infty(\Sigma) & k = 0, \\ 0 & k \geq 1. \end{cases}$$

The proof of this theorem is technical, and not very illuminating. The resolution degree of this resolution is called *antighost number* and we have

$$\text{antigh}\, \mathcal{P}_a = 1. \tag{3.53}$$

3.4 EXTENDED PHASE SPACE

As we showed previously, the longitudinal forms along the gauge orbits can be represented as polynomials in the 1-forms η^a, dual to the vector fields defining the infinitesimal gauge transformations, with coefficients that are smooth functions on Σ. The exterior longitudinal algebra is therefore isomorphic to

$$\mathcal{C}^\infty(\Sigma) \otimes \mathbb{C}[\eta^a].$$

The generators η^a are called *the ghosts*, and the form degree is called *the pure ghost number*,

$$\text{pure gh}\, \eta^a = 1.$$

Recall that the action of d is actually well-defined on the algebra $\mathcal{C}^\infty(P) \otimes \mathbb{C}[\eta^a]$ because the gauge transformations are defined everywhere in P and not just the constraint surface Σ. It is not a differential in $\mathcal{C}^\infty(P) \otimes \mathbb{C}[\eta^a]$ however, its square is only weakly zero.

Remarkably, the Koszul-Tate complex and the exterior longitudinal complex possess the same number of generators, which furthermore share the same Grassmann parity. This is because the functions G_a determine both, the constraint surface (through $G_a = 0$) and the gauge transformations (through $\delta_\epsilon F = [F, G_a]\epsilon^a$). From now on we shall consider the tensor product algebra

$$\mathbb{C}[\mathcal{P}_a] \otimes \mathcal{C}^\infty(P) \otimes \mathbb{C}[\eta^a], \tag{3.54}$$

which we call *extended phase space* P_{ext}. We can extend the Poisson bracket structure to P_{ext} by declaring

$$\delta^a_b = [\eta^a, \mathcal{P}_b] = -[\mathcal{P}_b, \eta^a]. \tag{3.55}$$

The brackets of η^a and $\mathcal{P}_a$ with the original variables are taken to be zero. Because of (3.55) the generators $\mathcal{P}_a$ are called *the ghost momenta*. The antighost and pure ghost numbers are extended to P_{ext} by setting

$$\text{antigh}\, \eta^a = 0 \qquad \text{and} \qquad \text{pure gh}\, \mathcal{P}_a = 0.$$

On P_{ext} we can then define the *total ghost number*, or just *ghost number*, by

$$\text{gh}\, A := \text{pure gh}\, A - \text{antigh}\, A, \tag{3.56}$$

for any variable A of definite pure ghost and antighost number. The ghost number of ordinary functions is zero and

$$\operatorname{gh}\eta^a = 1 = -\operatorname{gh}\mathcal{P}_a. \tag{3.57}$$

Noticing

$$\operatorname{gh}(AB) = \operatorname{gh}A + \operatorname{gh}B, \tag{3.58}$$

we see that in a product, the ghosts count positively, while the ghost momenta count negatively towards the total ghost number. So far, the differentials δ and d are only defined on $\mathbb{C}[\mathcal{P}_a] \otimes \mathcal{C}^\infty(P)$ and $\mathcal{C}^\infty(P) \otimes \mathbb{C}[\eta^a]$ respectively. Because we want δ to have antighost number minus one, $\operatorname{antigh}\delta = -1$, we have to set $\delta\eta^a = 0$. While the requirements on d,

$$\operatorname{antigh} d = 0 \qquad \text{and} \qquad \operatorname{gh} d = 1,$$

only restrict $d\mathcal{P}_a$ to be linear in $\mathcal{P}$ and η, which would include simply taking $d\mathcal{P}_a = 0$, this choice would not make d anticommute with δ. Instead, setting

$$d\mathcal{P}_a = -\eta^c C_{ca}{}^b \mathcal{P}_b$$

yields the desired commutation relation $[\delta, d] = 0$.

3.5 THE BRST GENERATOR

Definition 3.5.1 (Differential modulo δ)**.** Let δ be a differential and d an odd derivation. We say that d is a *differential modulo* δ if

(i) d commutes with δ, i.e. $[d, \delta] = d\delta + \delta d = 0$. and

(ii) d^2 is δ exact, i.e. there exists a derivation Δ such that $d^2 = [\delta, \Delta]$.

The first condition implies that d induces a derivation in $H_\bullet(\delta)$. This derivation is still denoted d. The second condition expresses that the induced derivation is a differential, i.e. $d^2 = 0$ in $H_\bullet(\delta)$. If we take δ to be the Koszul-Tate differential, notice that the longitudinal derivative d is a differential modulo δ. We come now to an important theorem in BRST theory, that is sometimes called the *main theorem of homological perturbation theory*.

Theorem 3.5.2 (Main Theorem)**.** Let δ be a differential that provides a homological resolution of $H_0(\delta)$. We denote the resolution degree with r. Let d be a differential modulo δ of resolution degree zero, i.e.

$$\begin{aligned} d^2 &= -[\delta, \overset{(1)}{s}], \\ r(d) &= 0, \\ \delta d + d\delta &= 0, \end{aligned}$$

where $\overset{(1)}{s}$ is a derivation of resolution degree one. The degree associated with d is denoted by deg, we assume deg to be unchanged by δ, and for definiteness take d to be a coboundary operator, i.e.

$$\deg\delta = 0, \quad \deg d = 1.$$

We also define another grading

$$\mathrm{gh}(x) = \deg(x) - r(x),$$

which is called *total ghost number*, for obvious reasons, and we have $\mathrm{gh}\,\delta = \mathrm{gh}\,d = 1$. The main theorem now combines d with δ into a single differential.

(i) If for all derivations Δ with resolution degree $r(\Delta) \geq 1$ we have

$$[\Delta, \delta] = 0 \quad \Rightarrow \quad \Delta = [\widetilde{\Delta}, \delta],$$

then there exists a differential s of total ghost number 1 that combines δ with d,

$$s = \delta + d + \overset{(1)}{s} + \overset{(2)}{s},$$

that satisfies $s^2 = 0$, and such that

$$r(\overset{(k)}{s}) = k \qquad \text{and} \qquad \mathrm{gh}(\overset{(k)}{s}) = 1.$$

(ii) Any differential s that combines δ with d in this way fulfills $H^k(s) = H^k(d)$, where the cohomology of d is computed in $H_0(\delta)$, i.e. is defined by the equations

$$dx = \delta y, \qquad x \sim x + dz + \delta z',$$

with

$$r(x) = r(z) = 0 \qquad \text{and} \qquad r(y) = r(z') = 1.$$

In our specific case, where δ is the Koszul-Tate differential and d is the longitudinal derivative, the main theorem of homological perturbation theory therefore guarantees the existence of the BRST differential

$$s = \delta + d + \ldots$$

such that $s^2 = 0$, where the dots indicate further terms of antighost number greater than zero. Furthermore, it turns out that s can be chosen to have a canonical action, i.e. there exists a function $\Omega \in P_{\mathrm{ext}}$, called the *BRST function*, such that

$$sF = [F, \Omega].$$

The nilpotency of s is equivalent to

$$[\Omega, \Omega] = 0,$$

by virtue of the Jacobi identity, This is a nontrivial property because the grading properties of s imply that Ω is Grassmann odd, which means this graded commutator is symmetric. To lowest antighost order, Ω is always given by

$$\Omega = \eta^a G_a + \ldots,$$

where the dots indicate further terms that contain at least one ghost momentum P_a and hence at least two η^a's.

3.6 DYNAMICS OF THE GHOSTS

We have established that the cohomology $H^\bullet(s)$ of the BRST differential s is isomorphic to the cohomology of the longitudinal exterior derivative at ghost number $g \geq 0$ and vanishes for $g < 0$,

$$H^g(s) = \begin{cases} 0, & g < 0, \\ H^g(d), & g \geq 0. \end{cases}$$

In particular we have

$$H^0(s) = \{\text{classical observables}\},$$

where with classical observables we mean the equivalence classes of weakly gauge invariant functions that coincide on the constraint surface. Because of this one calls *BRST observables* the ghost number zero functions $A(q, p, \eta, \mathcal{P})$ which are BRST invariant,

$$\begin{aligned} sA \equiv [A, \Omega] &= 0, \\ \text{gh}\, A &= 0, \end{aligned}$$

with the understanding that two BRST observables that differ by a BRST-exact one should be identified. The connection between a BRST observable $A(q, p, \eta, \mathcal{P})$ and the corresponding classical observable $A_0(q, p)$ is simply that the antighost number zero component $\overset{(0)}{A}$ of A is equal to A_0, i.e.

$$A(q, p, \eta, \mathcal{P}) = A_0(q, p) + \sum_{a \geq 1} \overset{(a)}{A}(q, p, \eta, \mathcal{P}),$$

where the terms of antighost number $a \geq 1$ are determined recursively in analogy to the proof of the main theorem. Because of this one says that A is a *BRST-invariant extension* of A_0. In order to construct a physical theory, one needs, in addition to the set of physically relevant functions, a bracket structure among them, and a Hamiltonian to generate the motion. The bracket structure then determines the graded commutation rules in the quantum theory. If follows that

(i) the set $H^0(s)$ of BRST observables should be endowed with a bracket structure.

(ii) the isomorphism between BRST observables and gauge invariant functions on Σ should preserve this bracket structure.

Luckily the Poisson bracket on P_{ext} induces just such a structure, i.e.

(i) if A and B are BRST-observables, then AB and $[A, B]$ are also BRST-observables, and if A' and B' are in the same cohomological classes as A and B respectively, then $A'B'$ and $[A', B']$ are in the same cohomological classes as AB and $[A, B]$ respectively.

This follows directly from the fact that s is a canonical transformation.

(ii) if A and B are BRST-invariant extensions of A_0 and B_0 respectively, then AB and $[A, B]$ are BRST-invariant extensions of A_0B_0 and $[A_0, B_0]$ respectively.

Let $Q_0(q, p)$ be the generator of a global symmetry. The transformation generated by Q_0 is only defined on the original phase space. How should it be extended to the

ghosts? In a relativistic gauge theory for instance, in which way should the ghosts transform under Poincaré transformations?

Having established the formalism, the answer to this question is now almost obvious. Since, as it turns out, the symmetry generator Q_0 is gauge invariant, it should be extended in a BRST-invariant way,

$$Q_0 \longrightarrow Q = Q_0 + \text{ghost terms},$$

such that

$$[Q, \Omega] = 0 \quad \text{and} \quad \text{gh}\, Q = 0.$$

Once Q has been determined, the ghost transformation properties simply follow from taking their bracket with Q. Furthermore, the transformation properties of the original variables generally get modified by ghost contributions induced by the terms that are added to Q_0 in order to form a BRST-invariant operator.

3.7 GAUGE FIXING

In the original phase space the time evolution is generated by the extended Hamiltonian $H_0 + \lambda^a G_a$. Because $H_0(q, p)$ is gauge invariant,

$$[H_0, G_a] = V^b{}_a\, G_b,$$

it possesses a BRST-invariant extension H, i.e. $[H, \Omega] = 0$. Consider the time evolution in P_{ext} generated by H

$$\dot{F} = [F, H]. \tag{3.59}$$

If F is the BRST-invariant extension of the gauge invariant function $F_0(q, p)$, then this evolution is equivalent to $\dot{F}_0 \approx [F_0, H_0]$, because $[F_0, G_a] \approx 0$ and the bracket is preserved under the correspondence that associates a BRST-invariant function to a gauge invariant one. The BRST-invariant Hamiltonian may be used to define the time evolution of all variables in P_{ext} regardless of whether they are gauge invariant or not. In particular, for the ghosts one finds

$$\dot{\eta}^a = [\eta^a, H] = \eta^b V_b{}^a + \ldots,$$
$$\dot{\mathcal{P}}_a = [\mathcal{P}_a, H] = -V_a{}^b \mathcal{P}_b + \ldots,$$

where $\ldots$ are terms of higher antighost number. We say that the equations of motion (3.59) are *gauge fixed* because they yield a non-ambiguous time evolution in P_{ext} with no arbitrariness. If one changes the BRST-invariant extension of the Hamiltonian by adding an Ω-exact term,

$$H \longrightarrow H + [K, \Omega], \tag{3.60}$$

one does not modify the dynamics of BRST-invariant functions at the cohomological level. However, the time evolution of the ghosts and of other non-BRST-invariant functions is generally changed. The function K is called *gauge fixing fermion* for this reason. Notice that consistency requires K to be Grassmann odd. The transformation (3.60) replaces the gauge invariant Hamiltonian H_0 by $H_0 + k^a G_a$, where $k^a \mathcal{P}_a$ is the term of antighost number one in K.

We can now see in what sense the BRST transformation is a symmetry transformation of the theory. Its Poisson bracket with the gauge fixed Hamiltonian $H + [K, \Omega]$ always vanishes, no matter which gauge fixing fermion K is chosen. The gauge fixed dynamics accordingly are always gauge invariant. The canonical action from which the dynamics in P_{ext} follow is called the *gauge fixed action* and is given by

$$S_K[q, p, \eta, \mathcal{P}] = \int_{t_1}^{t_2} \left(\dot{q}^i p_i + \dot{\eta}^a \mathcal{P}_a - H - [K, \Omega] \right) dt.$$

Because the Hamiltonian is BRST invariant and the BRST transformation is canonical, which therefore leaves the kinetic term invariant up to a boundary term at t_1 and t_2, the action S_K is also BRST invariant, up to that same boundary term. We stress that the choice of gauge fixed Hamiltonian H and gauge-fixed action comes *after* the determination of the BRST symmetry. In P_{ext} the BRST symmetry is intrinsic and takes the same form no matter what K is in S_K.

CHAPTER 4

JET BUNDLES

It turns out that jet bundles are the natural framework in which to discuss classical field theories. For the following topics we only require the very basic notions, nonetheless we will spend significant time discussing these basic topics to become familiar with the underlying concepts. This chapter is based on the very good book [Sau89] by D. J. Saunders. To those entirely unfamiliar with the language of fiber bundles, we recommend simply skipping this chapter entirely. The remaining chapters will still be intelligible without an intimate understanding of jet bundles, and at least the language will appear familiar to anyone who has ever dealt with field theories.

4.1 THE VERTICAL BUNDLE AND THE TRANSVERSE BUNDLE

We assume that the reader is familiar with the basic notions of fiber bundles, which will be our jumping off point. Before moving to the definition of jet bundles proper, there are several topics we have to address first. We begin by defining vertical vector fields and horizontal forms.

Definition 4.1.1 (Vertical Bundle)**.** Let $\pi\colon E \to M$ be a fiber bundle. The kernel of the tangent mapping $T\pi\colon TE \to TM$ is called the *vertical bundle* $VE := \ker T\pi$ of E.

Remark. Since $T_u\pi$ is surjective at every point $u \in E$, this construction yields a regular subbundle of TE. Furthermore, VE is integrable (without proof).

Definition 4.1.2 (Vertical Tangent Space)**.** Let $\pi\colon E \to M$ be a fiber bundle and denote with E_x the fiber over $x \in M$. The *vertical tangent space* V_uE of E in the point $u \in E_x$ is defined as the tangent space $T_u(E_x)$ to the fiber.

Remark. Keep in mind that via the natural inclusion of the tangent space to a submanifold we have $V_uE \subseteq T_uE$.

Lemma 4.1.3. The vertical tangent space V_uE at a point $u \in E_x$ coincides with the fiber of the vertical bundle $(VE)_u = \ker T_u\pi$.

Proof. Since $\pi|_{E_x} = x$ is constant on E_x, we get $V_uE \subseteq \ker(T_u\pi)$ and because π is a

submersion, $T_u\pi$ is surjective and therefore

$$T_uE/(VE)_u \cong T_{\pi(u)}M = T_xM.$$

This implies

$$\dim\left(\ker T_p\pi\right) = \dim E - \dim M = \dim E_x,$$

from which the claim follows. □

Definition 4.1.4 (Vertical Vector field)**.** A vector field $\xi \in \mathfrak{X}(E)$ is called *vertical* if $\xi_u \in V_uE$ for all $u \in E$.

Definition 4.1.5 (Horizontal Form)**.** A differential form $\omega \in \Omega(E)$ is called *horizontal* if $i_\xi\,\omega = 0$ for all vertical vector fields $\xi \in \mathfrak{X}(E)$.

The concept of horizontal differential forms is dual to the concept of vertical vector fields. Before we start to introduce more structure let us briefly discuss another construction which makes sense on any fiber bundle. For this remember the following simple fact from linear algebra:

Lemma 4.1.6. Given a finite dimensional vector space V and a subspace $W \subseteq V$ we have

$$W^* \simeq V^*/W^0,$$

where $W^0 \subseteq V^*$ denotes the *annihilator* of W in V, $W^0 \coloneqq \{\, f \in V^* \mid f|_W = 0 \,\}$.

Proof. Let $\imath\colon W \to V$ be the inclusion map, then the dual map $\imath^*\colon V^* \to W^*$ is surjective with $\ker \imath^* = W^0$. □

If we consider the vertical space at a point $u \in E$ as a subspace of the total tangent space at that point

$$V_uE \equiv T_uE_x \subseteq T_uE,$$

then the preceding lemma gives

$$T_u^*V \simeq T_u^*E/(T_uV)^0,$$

which in turn implies

$$\Lambda^k\left(T_u^*V\right) \simeq \Lambda^k\left(T_u^*E/(T_uV)^0\right).$$

Definition 4.1.7 (Pullback Bundle)**.** Let $\pi\colon E \to N$ be a bundle and $f\colon M \to N$ a smooth map. The *pullback* of E along f is defined as

$$f^\star E \coloneqq \{(x,a) \in M \times E \mid f(x) = \pi(a)\}.$$

The fiber $(f^\star E)_y$ is canonically identified with the fiber $E_{f(y)}$ by the map $(x,a) \mapsto a$. Since $f^\star E$ is a subset of $M \times E$, its tangent space at (x,a) is given by:

$$T_{(x,a)}f^\star E = \{(X,V) \in T_xM \times T_aE \mid f_\star X = \pi_\star V\}.$$

Any section s of E induces a section of $f^\star E$ called the *pullback section*

$$f^\star s \coloneqq s \circ f.$$

Definition 4.1.8 (Transverse Bundle)**.** The *transverse bundle* to $\pi\colon E \to M$ is the pullback vector bundle $(\pi^\star(TM), \pi^\star(\tau_M), E)$, where for any manifold M we denote its tangent bundle by $\tau_M\colon TM \to M$.

Lemma 4.1.9. The sequence of vector bundle morphisms

$$0 \longrightarrow V\pi \longrightarrow TE \longrightarrow \pi^\star(TM) \longrightarrow 0$$

is exact, where the map $V\pi \to TE$ is the inclusion and the map $TE \to \pi^\star(TM)$ is given by $\xi \mapsto (\pi_\star\xi, \tau_E(\xi))$. Both maps cover the identity on E.

Proof. The sequence is exact at $V\pi$ because $V\pi \to TE$ is an inclusion. The map $TE \to \pi^\star(TM)$ is surjective because in local coordinates if $(\eta, a) \in \pi^\star(TM)$, with $a \in E$ and $\eta \in T_{\pi(a)}M$, we can write

$$\eta = \eta^i \frac{\partial}{\partial x^i}\bigg|_{\pi(a)}$$

and simply set

$$\xi := \eta^i \frac{\partial}{\partial x^i}\bigg|_{a} \in TE.$$

Exactness now follows from the fact that for $\xi \in TE$ we have

$$\xi \in V\pi \quad \Leftrightarrow \quad \pi_\star\xi = 0 \in T_{\pi(\tau_E(\xi))}M \quad \Leftrightarrow \quad (\pi_\star\xi, \tau_E(\xi)) = 0 \in \pi^\star(TM)_{\tau_E(\xi)}. \qquad \square$$

Remark. This sequence does not split in general. This corresponds to the fact that there is no distinguished subbundle of τ_E which complements the vertical bundle.

Definition 4.1.10 (Vertical Cotangent Bundle)**.** The *vertical cotangent bundle* to π is denoted $(V^*\pi, (\tau_E\big|_{V\pi})^*, E)$ and is defined to be the vector bundle dual to $(V\pi, \tau_E\big|_{V\pi}, E)$.

Remark. The bundle $(\tau_E\big|_{V\pi})^*$ is generally not the pullback bundle of a bundle over some manifold.

Definition 4.1.11 (Horizontal Cotangent Bundle)**.** The *cotangent bundle* to E *horizontal* over π is defined to be the pullback bundle $(\pi^\star(T^*M), \pi^\star(\tau_M^*), E)$, which can be identified with the subbundle $(\pi^\star(T^*M), \tau_E^*\big|_{\pi^\star(T^*M)}, E)$ of τ_E^*, where for any manifold M, we denote its cotangent bundle by $\tau_M^*\colon T^*M \to M$.

Lemma 4.1.12. The bundle $(\pi^\star(T^*M), \pi^\star(\tau_M^*), E)$ is isomorphic to the vector bundle dual to $(\pi^\star(TM), \pi^\star(\tau_M), E)$.

Proof. Let $a \in E$, $(\eta, a) \in \pi^\star(T^*M)_a$ and $(\xi, a) \in \pi^\star(TM)_a$, then $\eta \in T^*_{\pi(a)}M$ and $\xi \in T_{\pi(a)}M$, so the duality relationship between $\pi^\star(T^*M)_a$ and $\pi^\star(TM)_a$ can be obtained by the isomorphisms

$$\pi^\star(T^*M)_a \simeq T^*_{\pi(a)}M, \qquad\qquad \pi^\star(TM)_a \simeq T_{\pi(a)}M.$$

The duality of the fibers then implies the duality of the bundles. $\square$

Lemma 4.1.13. The sequence of vector bundle morphisms

$$0 \longrightarrow \pi^\star(T^*M) \longrightarrow T^*E \longrightarrow V^*\pi \longrightarrow 0$$

is exact, where the map $\pi^\star(T^*M) \to T^*E$ is given by $(\eta, a) \mapsto \pi^\star\eta \in T^*_aE$, and the map $T^*E \to V^*\pi$ is the transpose of the inclusion $V\pi \to TE$.

Proof. This follows by duality from Lemma 4.1.9, because $(\eta, a) \mapsto \pi^\star\eta$ is just the transpose of $\xi \mapsto (\pi_\star\xi, \tau_E(\xi))$. □

Lemma 4.1.14. The vector subbundle $(\pi^\star(T^*M), \tau_E^*\big|_{\pi^\star(T^*M)}, E)$ is the annihilator in τ_E^* of $\tau_E\big|_{V\pi}$. The vector subbundle $(V\pi, \tau_E\big|_{V\pi}, E)$ is the annihilator in τ_E of $\tau_E^*\big|_{\pi^\star(T^*M)}$.

Proof. Let $\eta \in \pi^\star(T^*M)_a$, the there exits $\alpha \in T^*_{\pi(a)}M$ such that $\eta = \pi^\star\alpha$; if $\xi \in V_a\pi$ then

$$\eta(\xi) = (\pi^\star\alpha)(\xi) = \alpha(\pi_\star\xi) = \alpha(0) = 0,$$

which implies $\pi^\star(T^*M)_a \subseteq (V_a\pi)^0$. Since

$$\dim V_a\pi = \dim T_a(E_{\pi(a)}) = \dim E - \dim M$$

and

$$\dim \pi^\star(T^*M)_a = \dim T^*_aM = \dim M,$$

we get

$$\dim(V_a\pi)^0 = \dim T_aE - \dim V_a\pi = \dim M = \dim \pi^\star(T^*M)_a$$

and so $\pi^\star(T^*M)_a \simeq (V_a\pi)^0$. The second half of the result is obtained by duality. □

4.2 FIRST-ORDER JET BUNDLES

Definition 4.2.1 (1-jets)**.** Let (E, π, M) be a bundle and $p \in M$. We say that two local sections $\phi, \psi \in \Gamma_p(\pi)$ are *1-equivalent at p* if

(i) $\phi(p) = \psi(p)$ and

(ii) $T_p\phi = T_p\psi$ as maps $T_p\phi\colon T_pM \to T_pE$.

The equivalence class containing ϕ is called the *1-jet of ϕ at p* and is denoted by $j^1_p\phi$.

Remark. In adapted coordinates (x^μ, u^A) around $\phi(p)$ the condition $T_p\phi = T_p\psi$ reads

$$\frac{\partial\phi^A}{\partial x^\mu} = \frac{\partial\psi^A}{\partial x^\mu}. \tag{4.1}$$

Definition 4.2.2 (First Jet Bundle)**.** The *first jet bundle* of π is the set

$$J^1\pi := \left\{ j^1_p\phi \;\middle|\; p \in M, \phi \in \Gamma_p(\pi) \right\}$$

The functions π_1 and $\pi_{1,0}$, called *source projection* and *target projection* respectively, are defined by

$$\begin{aligned} \pi_1\colon J^1\pi &\longrightarrow M \\ j^1_p\phi &\longmapsto p \end{aligned} \qquad\qquad \begin{aligned} \pi_{1,0}\colon J^1\pi &\longrightarrow E \\ j^1_p\phi &\longmapsto \phi(p). \end{aligned}$$

Remark. Having defined these projections, we see that $J^1\pi$ can be viewed as a bundle over M or alternatively as a bundle over E.

Definition 4.2.3. Let (E, π, M) be a bundle and (U, u) an adapted coordinate system on E, with $u = (x^\mu, u^A)$. The induced coordinate system (U^1, u^1) on $J^1\pi$ is defined by

$$U^1 := \left\{ j^1_p\phi \,\middle|\, \phi(p) \in U \right\},$$
$$u^1 := (x^\mu, u^A, u^A_\mu),$$

where $x^\mu(j^1_p\phi) = x^\mu(p)$, $u^A(j^1_p\phi) = u^A(\phi(p))$ and the nD new functions $u^A_\mu \colon U^1 \to \mathbb{R}$, specified by

$$u^A_\mu(j^1_p\phi) = \left.\frac{\partial \phi^A}{\partial x^\mu}\right|_p$$

are called *derivative coordinates*.

Proposition 4.2.4. Given an atlas of adapted charts (U, u) on E, the corresponding collection of (U^1, u^1) is a finite dimensional C^∞ atlas on $J^1\pi$. The topology induced by this atlas makes $J^1\pi$ into a smooth manifold.

The proof of this proposition is rather technical and therefore we omit it here. The suspicious reader may find it in Saunders [Sau89] Chapter 4.1. The fact that $J^1\pi$ is indeed a smooth manifold will also be a corollary of Theorem 4.2.7.

Lemma 4.2.5. The functions π_1 and $\pi_{1,0}$ are smooth surjective submersions.

Proof. $\pi_{1,0}$ is surjective because for each $a \in E$ there is always a local section ϕ of π such that $\phi(\pi(a)) = a$ and so $\pi_{1,0}(j^1_p\phi) = a$.

It is smooth at every $j^1_p\phi \in J^1\pi$ because using adapted coordinate charts (U, u) around $\pi_{1,0}(j^1_p\phi) = \phi(p)$ and (U^1, u^1) around $j^1_p\phi$, the composite map $u \circ \pi_{1,0} \circ (u^1)^{-1}$ is just the projection

$$\mathrm{pr}_1 \colon \mathbb{R}^{n+D} \times \mathbb{R}^{nD} \supseteq u^1(U^1) \longrightarrow u(U) \subseteq \mathbb{R}^{n+D}.$$

It is a submersion for the same reason. This then also immediately implies the same result for $\pi_1 = \pi \circ \pi_{1,0}$. □

It turns out that $\pi_{1,0}$ has a natural structure as an affine bundle. The associated vector bundle is the tensor product $\pi^\star(T^\star M) \otimes V\pi$, formally the bundle

$$\left(\pi^\star(T^\star M) \otimes V\pi, \tau^\star_E\big|_{\pi^\star(T^\star M)} \otimes \tau_E\big|_{V\pi}, E\right). \tag{4.2}$$

Definition 4.2.6 (Affine Bundles)**.** Let (E, τ, M) be a vector bundle. An *affine bundle* modeled on τ is a quadruple (A, π, M, α) such that:

(i) (A, π, M) is a bundle.

(ii) for each $p \in M$ the map $\alpha \colon A \times_M E \to A$ satisfies $\alpha(A_p \times E_p) \subseteq A_p$ and $(A_p, E_p, \alpha\big|_{A_p \times E_p})$ is an affine space.

(iii) for every $p \in M$ there is a local trivialization $(U_p, \mathbb{R}^n, t_p)$ called an *affine local trivialization*, satisfying the condition that for each $q \in U_p$, the composite of

$$t_p\big|_{A_q} \colon A_q \longrightarrow \{q\} \times \mathbb{R}^n$$

with $\mathrm{pr}_2 \colon U_p \times \mathbb{R}^n \to \mathbb{R}^n$ is an affine isomorphism.

Theorem 4.2.7. The triple $(J^1\pi, pi_{1,0}, E)$ may be given the structure of an affine bundle, modeled on the vector bundle $\tau_E^\star\big|_{\pi^\star(T^\star M)} \otimes \tau_E\big|_{V\pi}$, in such a way that for each adapted chart (U, u) on E, the map

$$\begin{aligned} t_u \colon \pi_{1,0}^{-1}(U) &\longrightarrow \quad U \times \mathbb{R}^{nD} \\ j^1_p\phi \quad &\longmapsto \left(\phi(p), u^A_\mu(j^1_p\phi)\right) \end{aligned}$$

is an affine local trivialization.

Proof. Define the fiberwise action of the vector bundle on $\pi_{1,0}$ by prescribing the effect of the action upon the derivative coordinates of a given 1-jet. Let $a \in E$ and (U, u) be an adapted coordinate chart around a. A typical element $\xi \in (\pi^\star(T^\star M) \otimes V\pi)_a$ may be written in coordinates as

$$\xi = \xi^A_\mu \left(dx^\mu \otimes \frac{\partial}{\partial u^A}\right)_a.$$

The action of ξ on $j^1_{\pi(a)}\phi$ is written as $\xi[j^1_{\pi(a)}\phi]$ and defined by

$$u^A_\mu(\xi[j^1_{\pi(a)}\phi]) = u^A_\mu(j^1_{\pi(a)}\phi) + \xi^A_\mu.$$

To see that this is well-defined consider another chart (y^ν, v^B) around a and remember the transformation properties of jet coordinates

$$u^A_\mu(j^1_p\phi) = \left(\frac{\partial u^A}{\partial y^\nu}\bigg|_a + \frac{\partial u^A}{\partial v^B}\bigg|_a v^B_\nu(j^1_{\pi(a)}\phi)\right) \frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\pi(a)}.$$

Since ξ transforms as a tensor

$$\xi = \xi^A_\mu \left(dx^\mu \otimes \frac{\partial}{\partial u^A}\right)_a = \xi^B_\nu \frac{\partial u^a}{\partial v^B}\bigg|_a \frac{\partial y^\nu}{\partial x^\mu}\bigg|_a \left(dx^\mu \otimes \frac{\partial}{\partial u^A}\right)_a,$$

we see that

$$u^A_\mu(j^1_{\pi(a)}\phi) + \xi^A_\mu = \left(\frac{\partial u^A}{\partial y^\nu}\bigg|_a + \frac{\partial u^A}{\partial v^B}\bigg|_a \left(v^B_\nu(j^1_{\pi(a)}\phi) + \xi^B_\nu\right)\right) \frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\pi(a)}$$

has the correct transformation properties and so

$$v^B_\nu(\xi[j^1_{\pi(a)}\phi]) = v^B_\nu(j^1_{\pi(a)}\phi) + \xi^B_\nu.$$

Notice that we used

$$\frac{\partial y^\nu}{\partial x^\mu}\bigg|_{\pi(a)} = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_a.$$

It remains to be shown that the t_u define an affine local trivialization. Each t_u is a diffeomorphism because it is just the composite map $(u^{-1} \times \mathrm{id}_{\mathbb{R}^{nD}}) \circ u^1$, and clearly $\mathrm{pr}_1 \circ t_u = \pi_{1,0}\big|_{U^1}$. Therefore the t_u indeed define a local trivialization. To show that it is affine let $a \in U$ and consider the map $t_{u,a} \colon \pi_{1,0}^{-1}(a) \to \mathbb{R}^{nD}$ defined by

$$t_{u,a} := \mathrm{pr}_2 \circ t_u\big|_{\pi_{1,0}^{-1}(a)}.$$

Because $t_{u,a} = (u_\mu^A)\big|_{\pi_{1,0}^{-1}(a)}$, it is an affine morphism, where the fiber $\pi_{1,0}^{-1}(a)$ has the structure of an affine space given by the vector bundle action and $\mathbb{R}^{nD}$ has its natural affine structure. □

Definition 4.2.8 (First Prolongation of Sections)**.** Let (E, π, M) be a bundle, $W \subseteq M$ an open submanifold and $\phi \in \Gamma_W(\pi)$ a local section. The *first prolongation of* ϕ is the section $j^1\phi \in \Gamma_W(\pi_1)$ defined by

$$j^1\phi(p) := j^1_p\phi,$$

for all $p \in W$.

Remark. Notice that $\pi_{1,0} \circ j^1\phi = \phi$ and therefore $j^1(\pi_{1,0} \circ j^1\phi) = j^1\phi$. It is easily seen that this relation characterizes prolongations, which is also the statement of the following lemma.

Lemma 4.2.9. If $\psi \in \Gamma_W(\pi_1)$, then there exists a local section $\phi \in \Gamma_W(\pi)$ satisfying $\psi = j^1\phi$ if, and only if, $\psi = j^1(\pi_{1,0} \circ \psi)$.

The coordinate representation of $j^1\phi$ is exactly what one would expect. Simply compute

$$u^A(j^1\phi(p)) = u^A(j^1_p\phi) = u^A(\phi(p)) = \phi^A(p)$$

and similarly

$$u^A_\mu(j^1\phi(p)) = u^A_\mu(j^1_p\phi) = \left.\frac{\partial \phi^A}{\partial x^\mu}\right|_p.$$

The coordinate representation of $j^1\phi$ is therefore

$$\left(\phi^A, \partial_\mu \phi^A\right).$$

The most general local section $\psi \in \Gamma_W(\pi_1)$ has coordinates (ψ^A, ψ^A_μ), where in general the functions ψ^A_μ have nothing to do with the functions ψ^A. We conclude this section by describing a more pedestrian approach to thinking about the first order jet bundle as an affine bundle. Recall, that the fiber of the pullback bundle

$$\pi^\star(T^\star M) = \{\, (a, \omega) \in E \times T^\star M \mid \pi(a) = \tau^\star_M(\omega) \,\}$$

over $a \in E_p$ allows the natural identification

$$\pi^\star(T^\star M)_a \simeq T^\star_{\pi(a)} M = T^\star_p M,$$

which implies that we can identify the fibers of the vector bundle with spaces of linear maps

$$\begin{aligned}(\pi^\star(T^\star M) \otimes V\pi)_a &\simeq T^\star_p M \otimes V_a\pi \\ &\simeq \mathrm{Hom}(T_pM, V_a\pi).\end{aligned}$$

Rewriting the fiber of the jet bundle over $a \in E_p$ as

$$\begin{aligned}(J^1\pi)_a &= \left\{ j^1_p\phi \,\middle|\, \phi \in \Gamma_p(\pi) : \phi(p) = a \right\} \\ &\simeq \left\{ \gamma \in \mathrm{Hom}(T_pM, T_aE) \,\middle|\, T\pi \circ \gamma = \mathrm{id}_{T_pM} \right\},\end{aligned}$$

we see that in this picture the pointwise action of the vector bundle is simply addition of linear maps.

Remark. Any $\gamma \in J^1_a E$ specifies a splitting

$$T_aE = \mathrm{im}\,\gamma \oplus V^a\pi$$

which means sections of $\pi_{1,0}$ can be viewed as connections of π.

4.3 PROLONGATION OF MORPHISMS

Definition 4.3.1 (Prolongation of Morphisms)**.** Let $\pi\colon E \to M$ and $\rho\colon F \to N$ be bundles, and let $(f, \bar{f})$ be a bundle morphism, with $\bar{f} \in \mathrm{Diff}(M, N)$. The *first prolongation of* $(f, \bar{f})$ is the map

$$j^1(f, \bar{f})\colon J^1\pi \longrightarrow J^1\rho$$

defined by

$$j^1(f, \bar{f})(j^1_p\phi) = j^1_{\bar{f}(p)}(\tilde{f}(\phi))$$

where $\tilde{f} := f \circ \phi \circ \bar{f}^{-1}\big|_{\bar{f}(\text{domain of } \phi)}$, for every section $\phi \in \Gamma_p(\pi)$

The following two lemmata are intuitively clear, for explicit proofs see e.g. Saunders [Sau89] Lemma 4.2.6 and 4.2.7.

Lemma 4.3.2. Both maps, $(j^1 f, f)\colon \pi_{1,0} \to \rho_{1,0}$ and $(j^1 f, \bar{f})\colon \pi_1 \to \rho_1$, are bundle morphisms.

Lemma 4.3.3. If $f\colon \pi \to \rho$ and $g\colon \rho \to \sigma$ are bundle morphisms covering diffeomorphisms, then

$$\begin{aligned}j^1(g \circ f) &= j^1 g \circ j^1 f, \\ j^1(\mathrm{id}_E) &= \mathrm{id}_{J^1\pi}.\end{aligned}$$

Using coordinates (x^μ, u^A, u^A_μ) on $J^1\pi$ and (y^ν, v^B, v^B_ν) on $J^1\rho$ the coordinate representation of $j^1 f$ is given by

$$\begin{aligned}y^\nu \circ j^1 f &= f^\nu, \\ v^B \circ j^1 f &= f^B, \\ v^B_\nu \circ j^1 f &= \left(\frac{\partial f^B}{\partial x^\mu} + u^A_\mu \frac{\partial f^B}{\partial u^A}\right)\left(\frac{\partial (\bar{f}^{-1})^\mu}{\partial y^\nu} \circ \bar{f}\right).\end{aligned}$$

4.4 HIGHER-ORDER JET BUNDLES

Using multi index notation we are able to formulate k-jets and the k-th Jet bundle in complete analogy to the first order case.

Definition 4.4.1 (k-Jets)**.** Let $\pi\colon E \to M$ be a bundle, $p \in M$, and $\phi, \psi \in \Gamma_p(\pi)$. We say that ϕ and ψ are equivalent to order k if $\phi(p) = \psi(p)$ and

$$\left.\frac{\partial^{|\alpha|}\phi^A}{\partial x^\alpha}\right|_p = \left.\frac{\partial^{|\alpha|}\psi^A}{\partial x^\alpha}\right|_p,$$

for all $1 \leq |\alpha| \leq k$ and $1 \leq A \leq n$. Denote the equivalence class containing ϕ as $j^k_p\phi$.

Definition 4.4.2 (k-th Jet Bundle)**.**

$$J^k\pi := \left\{ j^k_p\phi \;\middle|\; p \in M, \phi \in \Gamma_p(\pi) \right\}$$

The following projections are defined in the obvious way and define bundles

$$\pi_k\colon J^k\pi \longrightarrow M,$$
$$\pi_{k,0}\colon J^k\pi \longrightarrow E,$$
$$\pi_{k,j}\colon J^k\pi \longrightarrow J^j\pi.$$

4.5 THE HORIZONTAL COMPLEX

Definition 4.5.1 (Generalized Vector Field)**.** A *generalized vector field* on $J^k\pi$ *of order* $l \geq k$ is a section of the pullback bundle $\pi^\star_{l,k}(TJ^k\pi)$.

Remark. For $l = k$ this reduces to the definition of an ordinary vector field on $J^k\pi$.

Definition 4.5.2 (Evolutionary Vector Field)**.** An *evolutionary vector field* is a section of the pullback bundle $\pi^\star_k(V\pi)$. The set of all such vector fields is denoted $\mathrm{Ev}(J^k\pi)$. As usual we shall consider an evolutionary vector field $X \in \mathrm{Ev}(J^k\pi)$ as a map

$$X\colon J^k\pi \longrightarrow V\pi.$$

In local coordinates

$$X = X^A \frac{\partial}{\partial u^A},$$

where X^A are local functions on $J^k\pi$.

Remark. We can consider evolutionary vector fields as generalized vector fields on E of order k that are vertical over $\pi_{k,0}$.

Definition 4.5.3 (Total Differential Operators)**.** A *total differential operator* is an operator

$$P\colon \mathrm{Ev}(J^k\pi) \longrightarrow \Omega^{r,s}(J^l\pi)$$

which is locally of the form

$$P(X) = \sum_{|\alpha|=0}^{k} (d_\alpha X^A) P^\alpha_A,$$

where each P^α_A is a smooth form of type (r, s).

4.6 CONTACT STRUCTURE

Definition 4.6.1 (Holonomic Lift)**.** Let $\pi\colon E \to M$ be a fiber bundle, $p \in M, \phi \in \Gamma_p(\pi)$ and $\xi \in T_pM$. The *holonomic lift* of ξ by ϕ is defined to be

$$(\phi_\star \xi, j^1_p\phi) \in \pi^\star_{1,0}(TE).$$

Theorem 4.6.2. Let $j^1_p\phi \in J^1\pi$, there is a canonical decomposition of the vector space $\pi^\star_{1,0}(TE)_{j^1_p\phi}$ as a direct sum of two subspaces

$$\pi^\star_{1,0}(V\pi)_{j^1_p\phi} \oplus \phi_\star(T_pM),$$

where $\phi_\star(T_pM) \equiv \phi_\star(T_pM) \times \{j^1_p\phi\}$.

Proof. The jet $j^1_p\phi$ characterizes the map $\phi_\star$ completely, so $\phi_\star(T_pM) \subseteq \pi^\star_{1,0}(TE)_{j^1_p\phi}$ is well-defined. If $(\xi, j^1_p\phi) \in \pi^\star_{1,0}(TE)_{j^1_p\phi}$ then

$$(\phi_\star\pi_\star\xi, j^1_p\phi) \in \phi_\star(T_pM)$$

and

$$(\xi - \phi_\star\pi_\star\xi, j^1_p\phi) \in \pi^\star_{1,0}(V\pi)_{j^1_p\phi},$$

because $\pi_\star(\xi - \phi_\star\pi_\star\xi) = 0$. On the other hand, should

$$(\xi, j^1_p\phi) \in \pi^\star_{1,0}(V\pi) \cap \phi_\star(T_pM)$$

then there exists a $\eta \in T_pM$ such that $\xi = \phi_\star\eta$ and we know that $\pi_\star\xi = 0$. These two statements already imply $\eta = \pi_\star\phi_\star\eta = 0$ and therefore $\xi = 0$. □

Remark. Canonical refers to the lack of choices in this construction.

Corollary 4.6.3. The bundle $(\pi^\star_{1,0}(TE), \pi^\star_{1,0}(\tau_E), J^1\pi)$ may be written as a direct sum of two subbundles

$$(\pi^\star_{1,0}(V\pi) \oplus H\pi_{1,0}, \pi^\star_{1,0}(\tau_E), J^1\pi),$$

where $H\pi_{1,0}$ is the union of the fibers $\phi_\star(T_pM)$.

The dual concept to holonomic tangent vector is called contact cotangent vector. While a holonomic tangent vector lies in the image of $\phi_\star$ for some local section ϕ, a contact cotangent vector lies in in the kernel of $\phi^\star$.

Definition 4.6.4 (Contact Cotangent Vector)**.** An element $(\eta, j^1_p\phi) \in \pi^\star_{1,0}(T^*E)$ is called *contact cotangent vector*, or *contact covector*, if $\phi^\star\eta = 0$.

Remark. It is clear that the condition $\phi^\star\eta = 0$ does not depend on the particular choice of section ϕ but only on the jet $j^1_p\phi$.

We say that holonomic and contact elements are dual to each other because of the duality between $\pi^\star_{1,0}(\tau_E)$ and $\pi^\star_{1,0}(\tau^*_E)$, cf. Lemma 4.1.12, and the fact that the decomposition of $\pi^\star(\tau_E)$ into holonomic and vertical tangent vectors corresponds to the decomposition of $\pi^\star(\tau^*_E)$ into contact and horizontal elements, in the sense that contact and holonomic elements annihilate each other, as do horizontal and vertical elements. This statement is the subject of the following propositions.

Proposition 4.6.5. Let $j^1_p\phi \in J^1\pi$, then

$$\pi^\star_{1,0}(\pi^\star(T^*M))_{j^1_p\phi} = (\pi^\star_{1,0}(V\pi)_{j^1_p\phi})^0$$

and

$$\pi^\star_{1,0}(\ker\phi^\star) = (\phi_\star(T_pM))^0,$$

where $\pi^\star(T^*M)$ denotes the image of T^*M in T^*E under the pullback by π, and $\pi^\star_{1,0}(\ker\phi^\star)$ denotes the set of contact covectors in $\pi^\star_{1,0}(T^*E)_{j^1_p\phi}$.

Proof. The first statement is a direct consequence of Lemma 4.1.14. The second statement is proved in a very similar manner.
Let $(\eta, j^1_p\phi) \in \pi^\star_{1,0}(\ker\phi^\star)$ and $(\xi, j^1_p\phi) \in \phi_\star(T_pM)$. Then there exists a $\zeta \in T_pM$ such that $\xi = \phi_\star\zeta$, which means

$$\eta(\xi) = \eta(\phi_\star\zeta) = (\phi^\star\eta)(\zeta) = 0$$

and therefore $\pi^\star_{1,0}(\ker\phi^\star) \subseteq (\phi_\star(T_pM))^0$. To show equality simply note that

$$\dim\pi^\star_{1,0}(\ker\phi^\star) = \dim\ker\phi^\star = \dim E - \dim M$$

and

$$\dim\phi_\star(T_pM) = \dim T_pM = \dim M. \qquad \square$$

Theorem 4.6.6. Let $j^1_p\phi \in J^1\pi$, there is a canonical decomposition of the vector space $\pi^\star_{1,0}(T^*E)_{j^1_p\phi}$ as a direct sum

$$\pi^\star_{1,0}(\pi^\star(T^*M))_{j^1_p\phi} \oplus \pi^\star_{1,0}(\ker\phi^\star).$$

Proof. Using Proposition 4.6.5, this follows from Theorem 4.6.2 by duality. $\square$

Corollary 4.6.7. The bundle $(\pi^\star_{1,0}(T^*E), \pi^\star_{1,0}(\tau^*_E), J^1\pi)$ may be written as the direct sum of two subbundles

$$(\pi^\star_{1,0}(\pi^\star(T^*M)) \oplus C^*\pi_{1,0}, \pi^\star_{1,0}(\tau^*_E), J^1\pi),$$

where $C^*\pi_{1,0}$ is the union of the fibers $\pi^\star(\ker\phi^\star)$.

Definition 4.6.8 (Total Derivative)**.** We denote the space of vector fields along $\pi_{1,0}$, i.e. sections of $\pi^\star_{1,0}(\tau_E)$, by $\mathfrak{X}(\pi_{1,0})$. The submodule of $\mathfrak{X}(\pi_{1,0})$ corresponding to sections of $\pi^\star_{1,0}(\tau_E)\big|_{\pi^\star_{1,0}(V\pi)}$ is denoted by $\mathfrak{X}^v(\pi_{1,0})$, and the submodule corresponding to sections of $\pi^\star_{1,0}(\tau_E)\big|_{H\pi_{1,0}}$ is denoted by $\mathfrak{X}^h(\pi_{1,0})$. An element of $\mathfrak{X}^h(\pi_{1,0})$ is called *total derivative*.

The decomposition of the bundle $\pi^\star_{1,0}(\tau_E)$, cf. Corollary 4.6.3, directly implies the corresponding decomposition of $\mathfrak{X}(\pi_{1,0})$.

Corollary 4.6.9. The module $\mathfrak{X}(\pi_{1,0})$ can be written as the direct sum of its submodules

$$\mathfrak{X}(\pi_{1,0}) = \mathfrak{X}^v(\pi_{1,0}) \oplus \mathfrak{X}^h(\pi_{1,0}).$$

While vector fields along $\pi_{1,0}$ are actually maps $J^1\pi \to \pi_{1,0}^\star(TE)$, we shall usually consider them as maps $J^1\pi \to TE$. If $X \in \mathfrak{X}^h(\pi_{1,0})$, then in coordinates

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu} + u^A_\mu \frac{\partial}{\partial u^A} \right),$$

for some functions X^μ defined on $J^1\pi$. The total derivatives

$$d_\mu \equiv \frac{d}{dx^\mu} \coloneqq \frac{\partial}{\partial x^\mu} + u^A_\mu \frac{\partial}{\partial u^A}$$

are called *coordinate total derivatives*. For $f \in \mathcal{C}^\infty(E)$ the action of X on f yields a function $Xf \in \mathcal{C}^\infty(J^1\pi)$, which in coordinates is given by

$$Xf = X^\mu \frac{df}{dx^\mu} = X^\mu \left(\frac{\partial f}{\partial x^\mu} + u^A_\mu \frac{\partial f}{\partial u^A} \right).$$

Note in particular that

$$\frac{du^A}{dx^\mu} = u^A_\mu \in \mathcal{C}^\infty(J^1\pi).$$

We shall denote the module of sections of $\pi_{1,0}^\star(T^*E)$ by $\bigwedge^1_0 \pi_{1,0}$, which we regard as a submodule of $\bigwedge^1 J^1\pi$, using the identification of $\pi_{1,0}^\star(T^*E)$ as a submanifold of $T^*J^1\pi$. The module of sections of $\pi_{1,0}^\star(\tau_E^*\big|_{\pi^\star(T^*M)})$ is denoted $\bigwedge^1_0 \pi_1$.

Definition 4.6.10 (Contact Forms)**.** The module of sections of $\pi_{1,0}^\star(\tau_E^*)\big|_{C^*\pi_{1,0}}$ is denoted by $\bigwedge^1_C \pi_{1,0}$. An element of $\bigwedge^1_C \pi_{1,0}$ is called a *contact form*.

As a consequence of Corollary 4.6.7 we get the corresponding decomposition of $\bigwedge^1_0 \pi_{1,0}$.

Theorem 4.6.11. The module $\bigwedge^1_0 \pi_{1,0}$ can be written as the direct sum of two submodules

$$\bigwedge^1_0 \pi_{1,0} = \bigwedge^1_0 \pi_1 \oplus \bigwedge^1_C \pi_{1,0}.$$

To recap, we have shown that each of the pullback bundles $(\pi_{1,0}^\star(TE), \pi_{1,0}^\star(\tau_E), J^1\pi)$ and $(\pi_{1,0}^\star(T^*E), \pi_{1,0}^\star(\tau_E^*), J^1\pi)$ can be canonically written as direct sums of vector bundles over $J^1\pi$

$$\begin{aligned} \pi_{1,0}^\star(TE) &= \pi_{1,0}^\star(V\pi) \oplus H\pi_{1,0} \\ \pi_{1,0}^\star(T^*E) &= \pi_{1,0}^\star(\pi^\star(T^*M)) \oplus C^*\pi_{1,0}, \end{aligned}$$

where these decompositions are in a sense dual to each other. This decomposition directly leads to define several important bundle endomorphisms and sections of the bundles $\pi_{1,0}^\star(\tau_E)$ and $\pi_{1,0}^\star(\tau_E^*)$.

Definition 4.6.12. The vector bundle endomorphisms (h, id_E) and (v, id_E) of $\pi_{1,0}^\star(\tau_E)$ are defined by

$$\begin{aligned} h(\xi^h + \xi^v) &= \xi^h \\ v(\xi^h + \xi^v) &= \xi^v, \end{aligned}$$

where $\xi^h \in H\pi_{1,0}$ and $\xi^v \in \pi_{1,0}^\star(V\pi)$.

Definition 4.6.13. The vector bundle endomorphisms (h, id_E) and (v, id_E) of $\pi_{1,0}^\star(\tau_E^*)$ are defined by

$$h(\eta^h + \eta^v) = \eta^h$$
$$v(\eta^h + \eta^v) = \eta^v,$$

where $\eta^h \in \pi_{1,0}^\star(\pi^\star(T^*M))$ and $\eta^v \in C^*\pi_{1,0}$.

Definition 4.6.14. The vector-valued 1-forms h and v are sections of the tensor bundle $\pi_{1,0}^\star(\tau_E^*) \otimes \pi^\star(\tau_E)$ defined by

$$h_{j_p^1\phi}(\xi, \eta) = \eta(h(\xi))$$
$$v_{j_p^1\phi}(\xi, \eta) = \eta(v(\xi)),$$

where $\xi \in \pi_{1,0}^\star(TE)_{j_p^1\phi}$ and $\eta \in \pi_{1,0}^\star(T^*E)_{j_p^1\phi}$.

If one regards $\pi_{1,0}^\star(\tau_E^*) \otimes \pi^\star(\tau_E)$ as a subbundle of $\tau_{J^1\pi}^* \otimes \pi_{1,0}^\star(\tau_E)$, then the sections h and v can be regarded as vector valued forms along $\pi_{1,0}$, and in coordinates they can be written as

$$h = dx^\mu \otimes \frac{d}{dx^\mu}$$
$$v = (du^A - u^A_\mu dx^\mu) \otimes \frac{\partial}{\partial u^A}.$$

The information encoded in the contact forms on $J^1\pi$ is also carried by h and v (in their various guises), and so they are together called *contact structure* on π_1. Sections of π_1 which are prolongations can be characterized by considering the vector subbundle of $\tau_{J^1\pi}$ containing only tangent vectors to $J^1\pi$ that project to holonomic tangent vectors under $(\pi_{1,0\star}, \tau_{J^1\pi})$. We shall now generalize the ideas we just introduced to the affine bundle $(J^{k+1}\pi, \pi_{k+1,k}, J^k\pi)$. This yields two families of $d_\star$ type derivations, called the *horizontal* and *vertical differentials*

Definition 4.6.15 (Holonomic Lift)**.** Let $p \in M, \phi \in \Gamma_p(\pi)$ and $\xi \in T_pM$. The k^{th} *holonomic lift* of ξ by ϕ is defined to be

$$((j^k\phi)_\star\xi, j_p^{k+1}\phi) \in \pi_{k+1,k}^\star(TJ^k\pi).$$

Remark. This element of $\pi_{k+1,k}^\star(TJ^k\pi)$ is called the k^{th} holonomic lift because it involves the tangent vector to $J^k\pi$. Consequently the holonomic lift to $\pi_{1,0}^\star(TE) \simeq \pi_{1,0}^\star(TJ^0\pi)$ is called the *zeroth holonomic lift*.

Theorem 4.6.16. Let $j_p^{k+1}\phi \in J^{k+1}\pi$, there is a canonical decomposition of the vector space $\pi_{k+1,k}^\star(TJ^k\pi)_{j_p^{k+1}\phi}$ as a direct sum of two subspaces

$$\pi_{k+1,k}^\star(V\pi_k)_{j_p^{k+1}\phi} \oplus (j^k\phi)_\star(T_pM),$$

where $(j^k\phi)_\star(T_pM)$ denotes the set of k^{th} holonomic lifts of tangent vectors in T_pM by the section ϕ.

Proof. We will only need to show that $(j^k\phi)_\star(T_pM) \subseteq \pi^\star_{k+1,k}(TJ^k\pi)_{j_p^{k+1}\phi}$ is well-defined. The rest of the proof of Theorem 4.6.2 goes through unchanged. Let $\gamma\colon \mathbb{R} \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = \xi$, then $(j^k\phi)_\star\xi = (j^k\phi \circ \gamma)'(0)$, and

$$\frac{d}{dt}\bigg|_{t=0} u^A_\alpha \circ j^k\phi \circ \gamma = \frac{\partial^{|\alpha|+1}\phi^A}{\partial x^{\alpha+1_\mu}}\bigg|_p \frac{d\gamma^\mu}{dt}\bigg|_{t=0}$$

depends only on derivatives of ϕ up to order $k+1$. □

Corollary 4.6.17. The vector bundle $(\pi^\star_{k+1,k}(TJ^k\pi), \pi^\star_{k+1,k}(\tau_{J^k\pi}), J^k\pi)$ may be written as the direct sum of two subbundles

$$(\pi^\star_{k+1,k}(V\pi_k) \oplus H\pi_{k+1,k}, \pi^\star_{k+1,k}(\tau_{J^k\pi}), J^k\pi),$$

where $H\pi_{k+1,k}$ is the union of the fibers $(j_k\phi)_\star(T_pM)$.

In coordinates, the k^{th} holonomic lift of

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu}\bigg|_p$$

is given by

$$(j^k\phi)_\star\xi = \xi^\mu\left(\frac{\partial}{\partial x^\mu}\bigg|_{j_p^k\phi} + \sum_{|\alpha|=0}^{k} u^A_{\alpha+1_\mu}(j_p^{k+1}\phi)\frac{\partial}{\partial u^A_\alpha}\bigg|_{j_p^k\phi}\right).$$

Definition 4.6.18 (Contact Cotangent Vector)**.** An element $(\eta, j_p^{k+1}\phi) \in \pi^\star_{k+1,k}(T^*J^k\pi)$ is called *contact cotangent vector* if $(j^k\phi)^\star\eta = 0$.

Again, this is well-defined because $(j^k\phi)^\star$ depends only on the derivatives of ϕ up to order $k+1$.

Proposition 4.6.19. Let $j_p^{k+1}\phi \in J^{k+1}\pi$, then

$$\pi^\star_{k+1,k}(\pi^\star_k(T^*M))_{j_p^{k+1}\phi} = \left(\pi^\star_{k+1,k}(V\pi_k)_{j_p^{k+1}\phi}\right)^0$$

and

$$\pi^\star_{k+1,k}(\ker(j^k\phi)^\star) = \left((j^k\phi)_\star(T_pM)\right)^0,$$

where $\pi^\star_k(T^*M)$ is regarded as a submanifold of $T^*J^k\pi$, and $\pi^\star_{k+1,k}(\ker(j^k\phi)^\star)$ denotes the set of contact covectors in $\pi^\star_{k+1,k}(T^*J^k\pi)_{j_p^{k+1}\phi}$.

Theorem 4.6.20. Let $j_p^{k+1}\phi \in J^{k+1}\pi$, there is a canonical decomposition of the vector space $\pi^\star_{k+1,k}(T^*J^k\pi)_{j_p^{k+1}\phi}$ as a direct sum

$$\pi^\star_{k+1,k}(\pi^\star_k(T^*M))_{j_p^{k+1}\phi} \oplus \pi^\star_{k+1,k}(\ker(j^k\phi)^\star).$$

Corollary 4.6.21. The vector bundle $(\pi^\star_{k+1,k}(T^*J^k\pi), \pi^\star_{k+1,k}(\tau^*_{J^k\pi}), J^{k+1}\pi)$ can be written as the direct sum of the two subbundles

$$(\pi^\star_{k+1,k}(\pi^\star_k(T^*M)) \oplus C^*\pi_{k+1,k}, \pi^\star_{k+1,k}(\tau^*_{J^k\pi}), J^{k+1}\pi),$$

where $C^*\pi_{k+1,k}$ denotes the union of the fibers $\pi^\star_{k+1,k}(\ker(j^k\phi)^\star)$.

Definition 4.6.22 (Total Derivative)**.** The submodule of $\mathfrak{X}(\pi_{k+1,k})$ corresponding to sections of $\pi^{\star}_{k+1,k}(\tau_{J^k\pi})\big|_{\pi^{\star}_{k+1,k}(V\pi_k)}$ will be denoted by $\mathfrak{X}^v(\pi_{k+1,k})$, while the submodule corresponding to sections of $\pi^{\star}_{k+1,k}(\tau_{J^k\pi})\big|_{H\pi_{k+1,k}}$ will be denoted by $\mathfrak{X}^h(\pi_{k+1,k})$. An element of $\mathfrak{X}^h(\pi_{k+1,k})$ will be called *total derivative*.

Each vector field $X \in \mathfrak{X}(M)$ corresponds to a total derivative, its k^{th} holonomic lift $X^k \in \mathfrak{X}^h(\pi_{k+1,k})$, according to

$$X^k_{j^{k+1}_p\phi} := (j^k\phi)_{\star}X_p.$$

Lemma 4.6.23. Two holonomic lifts X^k and X^l are π-related.

Proof. For $k > l$ simply compute

$$\begin{aligned}\pi_{k,l\star}\left(X^k_{j^{k+1}_p\phi}\right) &= \pi_{k,l\star}\left((j^k\phi)_{\star}X_p\right)\\ &= (j^l\phi)_{\star}X_p\\ &= X^l_{j^{l+1}_p\phi}\\ &= X^l_{\pi_{k+1,l+1}(j^{k+1}_p\phi)}.\end{aligned}$$

□

In coordinates, if

$$X = X^{\mu}\frac{\partial}{\partial x^{\mu}},$$

then the coordinate representation of the k^{th} holonomic lift of X is given by

$$X^k = X^{\mu}\left(\frac{\partial}{\partial x^{\mu}} + \sum_{|\alpha|=0}^{k} u^A_{\alpha+1_{\mu}}\frac{\partial}{\partial u^A_{\alpha}}\right).$$

Remark. X^k on the left-hand side is a vector field along $\pi_{k+1,k}$, while X^{μ} on the right-hand side is a function lifted from M to $J^{k+1}\pi$.

Definition 4.6.24 (Coordinate Total Derivatives)**.** The holonomic lifts of the local vector fields $\partial/\partial x^{\mu}$ are called *coordinate total derivatives* and their coordinate representation is

$$d_{\mu} \equiv \frac{d}{dx^{\mu}} := \frac{\partial}{\partial x^{\mu}} + \sum_{|\alpha|=0}^{k} u^A_{\alpha+1_{\mu}}\frac{\partial}{\partial u^A_{\alpha}}. \tag{4.3}$$

Definition 4.6.25 (Local Functions)**.** Let $U \subseteq J^k\pi$ be an open subset of the k-th jet space of π. A function $f \in \mathcal{C}^{\infty}(U, \mathbb{R})$ is called *local function of order k*. Another common name is *differential function of order k*.

When acting on a local function of order k, the total derivative produces a local function of order $k+1$. In particular acting on the coordinate functions u^A_{α} on $J^k\pi$ produces

$$\frac{du^A_{\alpha}}{dx^{\mu}} = u^A_{\alpha+1_{\mu}},$$

as functions on $J^{k+1}\pi$. The coordinate representation (4.3) makes it clear that total derivatives commute

$$[d_\mu, d_\nu] = 0.$$

To ease notation we shall also extend the multi index notation to total derivatives

$$d_\alpha := \prod_{\mu=0}^{D-1} \left(d_\mu\right)^{\alpha(\mu)}.$$

The corresponding decomposition of $\bigwedge^1_0 \pi_{k+1,k}$ is somewhat simpler, because the submodule of sections of $\pi^\star_{k+1,k}(\tau^*_{J^k\pi})\big|_{\pi^\star_{k+1,k}(\pi^\star_k(T^*M))}$ may be identified with $\bigwedge^1_0 \pi_{k+1}$.

Definition 4.6.26 (Contact Forms)**.** We denote the submodule of $\bigwedge^1_0 \pi_{k+1,k}$ corresponding to sections of $\pi^\star_{k+1,k}(\tau^\star_{J^k\pi})\big|_{C^*\pi_{k+1,k}}$ by $\bigwedge^1_C \pi_{k+1,k}$. Its elements are called *contact forms*.

Thus, we get the decomposition

$$\bigwedge^1_0 \pi_{k+1,k} = \bigwedge^1_0 \pi_{k+1} \oplus \bigwedge^1_C \pi_{k+1,k}.$$

Notice that $\bigwedge^1_C \pi_{k+1,k}$ and $\mathfrak{X}^h(\pi_{k+1,k})$ annihilate each other and so do $\bigwedge^1_0 \pi_{k+1}$ and $\mathfrak{X}^v(\pi_{k+1,k})$. In local coordinates a contact form σ can be written as

$$\sigma = \sum_{|\alpha|=0}^{k} \sigma^\alpha_A (du^A_\alpha - u^A_{\alpha+1_\mu} dx^\mu).$$

Definition 4.6.27. Analogously to Definition 4.6.13, the vector bundle endomorphisms $(h, \mathrm{id}_{J^k\pi})$ and $(v, \mathrm{id}_{J^k\pi})$ of $\pi^\star_{k+1,k}(\tau_{J^k\pi})$ are defined as

$$\begin{aligned} h(\xi^h + \xi^v) &= \xi^h \\ v(\xi^h + \xi^v) &= \xi^v, \end{aligned}$$

where $\xi^h \in H\pi_{k+1,k}$ and $\xi^v \in \pi^\star_{k+1,k}(V\pi_k)$.

Definition 4.6.28. Analogously to Definition 4.6.13, the vector bundle endomorphisms $(h, \mathrm{id}_{J^k\pi})$ and $(v, \mathrm{id}_{J^k\pi})$ of $\pi^\star_{k+1,k}(\tau^*_{J^k\pi})$ are defined as

$$\begin{aligned} h(\eta^h + \eta^v) &= \eta^h \\ v(\eta^h + \eta^v) &= \eta^v, \end{aligned}$$

where $\eta^h \in \pi^\star_{k+1,k}(\pi^\star_k(T^*M))$ and $\eta^v \in C^*\pi_{k+1,k}$.

If we consider $\pi^\star_{k+1,k}(\tau^*_{J^k\pi})$ to be a subbundle of $\tau^*_{J^{k+1}\pi}$, cf. Lemma 4.1.13, then we can write $h + v = \pi_{k+1,k\star}$.

Definition 4.6.29. The vector valued 1-forms h and v are the sections of the bundle $\pi^\star_{k+1,k}(\tau^*_{J^k\pi}) \otimes \pi^\star_{k+1,k}(\tau_{J^k\pi})$ defined by

$$\begin{aligned} h_{j^{k+1}_p\phi}(\xi, \eta) &= \eta(h(\xi)) \\ v_{j^{k+1}_p\phi}(\xi, \eta) &= \eta(v(\xi)), \end{aligned}$$

for $\xi \in \pi^\star_{k+1,k}(TJ^k\pi)_{j^{k+1}_p\phi}$ and $\eta \in \pi^\star_{k+1,k}(T^*J^k\pi)_{j^{k+1}_p\phi}$.

Again, considering $\pi^\star_{k+1,k}(\tau^*_{J^k\pi}) \otimes \pi^\star_{k+1,k}(\tau_{J^k\pi})$ as a subbundle of $\tau^*_{J^{k+1}\pi} \otimes \pi^\star_{k+1,k}(\tau_{J^k\pi})$, we can see h and v as vector-valued 1-forms along $\pi_{k+1,k}$. In local coordinates they take the form

$$h = dx^\mu \otimes \frac{d}{dx^\mu}$$
$$v = \sum_{|\alpha|=0}^{k} \left(du^A_\alpha - u^A_{\alpha+1_\mu} dx^\mu \right) \otimes \frac{\partial}{\partial u^A_\alpha}.$$

The vector-valued forms h and v induce derivations of type $\imath_\star$ and $d_\star$, cf. Section 2.5, aptly denoted by $\imath_h, \imath_v, d_h$ and d_v.

Lemma 4.6.30. The relationship $h + v = \pi_{k+1,k\star}$ implies the useful identity

$$d_h + d_v = \pi^\star_{k+1,k} \circ d.$$

Proof. Since $d_h + d_v$ is a derivation along $\pi_{k+1,k}$ (this is easy to show) and derivations are uniquely specified by how they act on functions and 1-forms it suffices to check how $d_h + d_v$ acts on functions $f \in \mathcal{C}^\infty(J^k\pi)$ and 1-forms $\vartheta \in \bigwedge^1 J^k\pi$. First, notice that for 1-forms

$$(\imath_h + \imath_v)\vartheta = \pi^\star_{k+1,k}\vartheta,$$

which gives

$$(d_h + d_v)f = (\imath_h + \imath_v)df = \pi^\star_{k+1,k} df,$$

where we used that df is a 1-form and $\imath_h f = \imath_v f = 0$. Now let $X_0, X_1 \in \mathfrak{X}(J^{k+1}\pi)$, the definition of $\imath_h$ gives

$$(\imath_h d\vartheta)(X_0, X_1) = d\vartheta(hX_0, \pi_{k+1,k\star}X_1) - d\vartheta(hX_1, \pi_{k+1,k\star}X_0).$$

Together with the analogous result for $\imath_v$ this shows

$$((\imath_h + \imath_v)d\vartheta)(X_0, X_1) = 2(\pi^\star_{k+1,k}\, d\vartheta)(X_0, X_1),$$

which already gives

$$(d_h + d_v)\vartheta = \pi^\star_{k+1,k}\, d\vartheta.$$

So we have shown that $d_h + d_v$ and $\pi^\star_{k+1,k} \circ d$ agree on functions as well as on 1-forms, which, since both are derivations, implies that they are equal. □

Lemma 4.6.31. The horizontal and vertical differentials satisfy

$$d_h^2 = d_v^2 = 0.$$

Proof. Using local coordinates one shows that d_h^2 and d_v^2 vanish on $\mathcal{C}^\infty(J^k\pi)$. Since they are both derivations of type $d_\star$ this implies that they vanish on $\bigwedge J^k\pi$. □

Corollary 4.6.32. The horizontal and vertical derivatives anticommute, i.e.

$$d_h \circ d_v + d_v \circ d_h = 0$$

Proof. This follows from $(d_h + d_v)^2 = \pi^\star_{k+2,k} \circ d^2$. □

Finally, we finish this chapter by summarizing the action of the horizontal and vertical differentials on the coordinate functions and coordinate 1-forms:

$$
\begin{array}{rcrcr}
\imath_h(dx^\mu) & = & d_h x^\mu & = & dx^\mu \\
\imath_h(du^A_\alpha) & = & d_h u^A_\alpha & = & u^A_{\alpha+1_\mu} dx^\mu \\
 & & d_h(dx^\mu) & = & 0 \\
 & & d_h(du^A_\alpha) & = & dx^\mu \wedge du^A_{\alpha+1_\mu} \\
 & & & & \\
\imath_v(dx^\mu) & = & d_v x^\mu & = & 0 \\
\imath_v(du^A_\alpha) & = & d_v u^A_\alpha & = & du^A_\alpha - u^A_{\alpha+1_\mu} dx^\mu \\
 & & d_v(dx^\mu) & = & 0 \\
 & & d_v(du^A_\alpha) & = & du^A_{\alpha+1_\mu} \wedge dx^\mu
\end{array}
$$

CHAPTER 5

THE ANTIFIELD FORMALISM

The BRST construction we have discussed so far applies only to Hamiltonian systems with a finite number of degrees of freedom. Since we want to apply the BRST machinery to higher-spin gauge theories, field theories, most naturally described in terms of a Lagrangian action formulation, we shall need a modified BRST construction, applicable to systems with an infinite amount of degrees of freedom, and, if at all possible, also directly applicable in the Lagrangian formulation, as to avoid losing the manifest covariant formulation when translating the Lagrangian to a Hamiltonian.

The standard method to do this is known as the antifield approach to BRST symmetry. It formulates the various BRST constructions directly in terms of Lagrangian variables and yields the same gauge-fixed action as the Hamiltonian formalism; it is however less complete [HT92].

This chapter is not based on any reference in particular and should be interpreted as the authors best attempt to give a brief introduction to the topic, as he sees it. However, most of what the author knows about this subject he has learned from [HT92], [NO90], [BD18], [BBH00], [Sta97] and [GMS09], listed in increasing order of mathematical rigor.

5.1 LAGRANGIAN FIELD THEORIES

We begin by briefly describing some standard concepts of the Lagrangian formalism of field theories, in terms of jet bundles. For a compact summary of the relevant theory of jet bundles, as well as a reference to the notation used in this work see Chapter 4. For the standard reference of Lagrangian field theory in terms of jet bundles and the variational bi-complex see the unfinished book by Anderson [And92].

Let M be a smooth oriented manifold of dimension D with volume form $\Omega \in \bigwedge^D M$. The manifold M takes the role of spacetime. The fields, i.e. the dynamic variables of the theory, are local sections of a vector bundle $\pi\colon E \to M$ and the dynamics of these fields are described by a Lagrangian.

Definition 5.1.1 (Lagrangian)**.** A *k-th order Lagrangian density* on $\pi\colon E \to M$ is a function $\mathcal{L} \in \mathcal{C}^\infty(J^k\pi)$. The corresponding *Lagrangian form* $L \in \bigwedge_0^D J^k\pi$ is simply defined to be the product with the volume form $L \coloneqq \mathcal{L}\,\Omega$.

Remark. The notation is introduced in Chapter 4.

The dynamics are described by $\mathcal{L}$ in the sense that we say that a local section of π satisfies the equations of motion if it is an extremal of $\mathcal{L}$.

Definition 5.1.2. The local section $\phi \in \Gamma_W(\pi)$, where W is an open subset of M, is an *extremal* of $\mathcal{L}$ if

$$\left.\frac{d}{dt}\right|_{t=0} \int_C j^k(\psi_t \circ \phi)^\star L = 0, \tag{5.1}$$

whenever $C \subseteq W$ is a compact D-dimensional submanifold of M and whenever the vertical vector field $X \in V(\pi)$ has flow ψ_t and satisfies $X\big|_{\pi^{-1}(\partial C)} = 0$. Here j^k denotes the k-th jet prolongation and $(\quad)^\star$ denotes the pullback.

Lemma 5.1.3. The local section ϕ is an extremal of L if, and only if,

$$\int_C (j^k\phi)^\star d_{X^k} L = 0, \tag{5.2}$$

where X^k is the k^{th} holonomic lift of X.

Using a local coordinate representation, this variational problem can be reformulated in terms of Euler-Lagrange equations and for $D = 1$ or $k = 1$ a global formulation in terms of the *Cartan form* and the *Euler-Lagrange form* is possible. However, if $D > 1$ and $k > 1$ this aforementioned construction runs into problems, namely the Cartan form used in the construction of the Euler-Lagrange form is not unique anymore.

When considering the variational problem we shall therefore assume the existence of appropriate coordinate charts that cover a sufficiently large (whatever that means) region of M, and be content with a description in local coordinates. Now, as usual, the local dynamics are described by Euler-Lagrange equations

$$G_A := \frac{\delta \mathcal{L}}{\delta \phi^A} = 0, \tag{5.3}$$

where the ϕ^A are local bundle coordinates and δ denotes the coordinate Euler-Lagrange derivative (Definition 5.2.4). A more mathematically satisfactory, global description of the dynamics can be made in the algebraic setting of the *variational bicomplex*, as described in [And92] this, however, is beyond the scope of this work and it is not clear what benefit this global approach would yield regarding the results of this thesis.

5.2 LOCAL FUNCTIONALS

Functions $f \in \mathcal{C}^\infty(U \subseteq J^k\pi)$ are called *local functions*. Here locality does not refer to the subset $U \subseteq J^k\pi$, but the finite order of the jet bundle $J^k\pi$. For example, the Lagrangian density $\mathcal{L} \in \mathcal{C}^\infty(J^k\pi)$ is a local function despite being defined everywhere on the jet bundle $J^k\pi$. For $p \in M$, we can *evaluate* the local function $f \in \mathcal{C}^\infty(J^k\pi, \mathbb{R})$ at a local section $\phi \in \Gamma_p(\pi)$:

$$f\big|_\phi := f \circ j^k\phi \equiv f\left(x^\mu, \phi^A(x^\mu), \frac{\partial^{|\alpha|}\phi^A}{\partial x^\alpha}(x^\mu)\right) \equiv f(x^\mu, \phi^A, \phi^A_\alpha), \tag{5.4}$$

where locality of the section now really refers to $p \in M$, and we assume the existence of fiber charts in a sufficiently large neighborhood of p. Notice that this evaluation is just pullback along the appropriate jet prolongation. Evaluation of local functions gives a neat characterization of the total derivative

$$(d_\mu f)\big|_\phi \equiv \left.\frac{df}{dx^\mu}\right|_\phi = \frac{d}{dx^\mu}\left(f\big|_\phi\right), \tag{5.5}$$

where on the right-hand side, d/dx^μ is just the ordinary total derivative on $\mathbb{R}^D$. We have already discussed that sections, or more generally, local sections of $\pi\colon E \to M$ and jet coordinates thereof are the dynamic variables of the theory. We shall now turn our discussion towards the observables of our theory. In the most general setting the observables are just functionals $\Gamma(\pi) \to \mathbb{R}$ [1] For us, this much generality only introduces problems at this point and we shall restrict our attention to so called *local functionals*.

Definition 5.2.1 (Local Functional)**.** A *local functional* is a map $F\colon \Gamma_c(\pi) \longrightarrow \mathbb{R}$ where $\Gamma_c(\pi)$ denotes the space of sections that have either compact support or vanish on the boundary, that can be written in the form

$$F(\phi) = \int_M d^D x\, f\big|_\phi \equiv \int_M d^D x\, (j^k\phi)^\star f, \tag{5.6}$$

where f is a local function on $J^k\pi$.

Here, locality again refers to the finite order k and serves to implement the physical concept of locality. Notice that a local functional depends on the derivative coordinates of the local section only up to finite order. As is the case for completely general integral transforms and their kernels, the local function in (5.6) is not uniquely determined. For example, we can always add a total derivative to the integral kernel, which does not modify the functional. However, this already exhausts all possibilities, as the following lemma shows.

Lemma 5.2.2. Two local functionals F and G are equal if, and only if,

$$f = g + d_\mu j^\mu, \tag{5.7}$$

where f and g are the integral kernels of F and G respectively and the j^μ are local functions.

Lemma 5.2.2 and Lemma 5.2.3 are proven in the excellent introduction by Barnich and Del Monte [BD18], where a bit more detail is provided as well. It turns out that the Euler-Lagrange derivative (Def 5.2.4) gives us a neat characterization of local functions which can be written as total derivatives.

Lemma 5.2.3. The Euler-Lagrange derivative of a local function vanishes identically if, and only if, it is a total divergence, i.e.

$$\frac{\delta f}{\delta \phi^A} = 0 \quad \Leftrightarrow \quad f = d_\mu j^\mu \tag{5.8}$$

for some local functions j^μ.

[1] More precisely, the observables are functionals that are defined only on those local sections that satisfy the equations of motion, i.e. are on shell.

Definition 5.2.4 (Euler-Lagrange Derivative)**.** The *Euler-Lagrange derivative* of a local function f is defined as

$$\frac{\delta f}{\delta \phi^A} := \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} d_\alpha \frac{\partial^{|\alpha|} f}{\partial \phi^A_\alpha}. \tag{5.9}$$

Remark. Notice that we have written ϕ^A_α for the derivative coordinates instead of something like u^A_α, as we did in Chapter 4. This abuse of notation is very popular in the physics community.

These results give us the ability to introduce a completely algebraic definition of local functionals:

Definition 5.2.5 (Algebraic Local Functional)**.** An *algebraic local functional* is an equivalence class of local functions under the equivalence relation

$$f \sim g \quad \Leftrightarrow \quad \frac{\delta f}{\delta \phi^A} = \frac{\delta g}{\delta \phi^A}. \tag{5.10}$$

We will denote the space of all algebraic local functionals on $\pi\colon E \to M$ simply by $\mathrm{Loc}(\pi)$.

Remark. In the following, when we talk about local functionals we will almost always be referring to the algebraic kind.

5.3 ANTIFIELDS

Our first task in defining the BRST-construction for the field-theoretic case is carrying over an appropriately modified version of the Koszul-Tate resolution (Section 3.3). A k^{th}-order Lagrangian defines a subset $\Sigma \subseteq J^k(\pi)$, by the condition

$$\left.\frac{\delta \mathcal{L}}{\delta \phi^A}\right|_\psi = 0 \quad \Leftrightarrow \quad \mathrm{Im}\, j^k\psi \subseteq \Sigma, \tag{5.11}$$

for local sections $\psi \in \Gamma_c(\pi)$. We shall assume enough regularity on the Euler-Lagrange equations such that Σ is a smooth manifold and call it *stationary surface* or simply *shell*. The set of local functionals defined on sections with image in Σ is denoted by $\mathrm{Loc}(\Sigma)$. Introducing the *stationary ideal* $\mathcal{I} \subseteq \mathrm{Loc}(\pi)$, defined to be those local functionals that vanish when restricted to Σ, allows us to establish the algebra isomorphism

$$\mathrm{Loc}(\Sigma) \cong {}^{\mathrm{Loc}(\pi)}\!/_{\mathcal{I}}. \tag{5.12}$$

We again have to assume regularity conditions on the Euler-Lagrange equations such that $\mathcal{I}$ is generated by the local functions

$$d_\alpha \frac{\delta \mathcal{L}}{\delta \phi^A} \equiv d_\alpha G_A, \tag{5.13}$$

which means that any local functional that vanishes *on shell* (i.e. on Σ) can be written using only (derivatives of) the Euler-Lagrange equations as its integral kernel. Equivalently, any algebraic local functional that vanishes on shell is a (derivative of) Euler-Lagrange equation, modulo total derivatives. For more information on these regularity conditions see [HT92].

The idea, as in the finite dimensional case, is now to adjoin (in some sensible manner) additional variables in order to describe $\mathrm{Loc}(\Sigma)$ in terms of a homological resolution, and in the following also describe Noether identities and gauge orbits in a similar manner.

For each field[2] $\partial_\alpha \phi^A \equiv \phi^A_\alpha$ we introduce a new Grassmann-odd object $\partial_\alpha \phi^*_A$, called *anti-field*, by adjoining the free graded-commutative algebra over the generators $\partial_\alpha \phi^*_A$ to the algebra of local functionals:

$$\mathcal{K} := \mathrm{Loc}(\pi) \otimes \bigwedge \partial_\alpha \phi^*_A. \tag{5.14}$$

We are quite sloppy with our notation here and denote the graded algebra simply by its generators. Once we are ready to introduce the Batalin-Vilkovisky complex (of which the Koszul complex is only one part) in its entirety we will be more explicit in our notation. We can extend the total derivative to $\mathcal{K}$ via $d_\alpha \partial_\beta \phi^*_A = \partial_{\alpha+\beta} \phi^*_A$ and define an additional derivation δ on $\mathcal{K}$, which vanishes on $\mathrm{Loc}(\pi)$, by

$$\begin{aligned} \delta\colon \quad \mathcal{K} &\longrightarrow \mathcal{K} \\ \partial_\alpha \phi^*_A &\longmapsto d_\alpha G_A. \end{aligned}$$

The differential algebra $(\mathcal{K}, \delta)$ together with the grading anti-gh$(\partial_\alpha \phi^*_A) = 1$, is called *Koszul-complex*. The definition of $(\mathcal{K}, \delta)$ immediately yields $\mathrm{Loc}(\Sigma) \cong H^0(\mathcal{K}, \delta)$, however, due to (possible) Noether identities, the higher homology groups do not vanish in general, so we have not yet constructed a resolution (Def 2.3.9). The simplest type of Noether identity is just an algebraic relation between the equations of motion

$$N^A G_A = 0, \tag{5.15}$$

where $N^A \in \mathrm{Loc}(\pi)$. In general, since we assumed that the stationary ideal is generated by the functions $d_\alpha G_A$, cf. (5.13), all *Noether identities* have to take the form

$$N^{A\alpha} d_\alpha G_A = 0, \tag{5.16}$$

where $N^{A\alpha} \in \mathrm{Loc}(\pi)$. Not all possible relations are of interest however. Notice that for $Q^{A\alpha B\beta} \in \mathrm{Loc}(\pi)$, anti-symmetric in the index pairs $A\alpha$ and $B\beta$, the following relation holds, just by the fact that we are considering a commutative algebra of functions:

$$Q^{A\alpha B\beta} d_\alpha G_A \, d_\beta G_B = 0. \tag{5.17}$$

Relations of this form are called *trivial* Noether identities, because they exist for any theory, independent of the explicit form of the equations of motion. For this reason trivial Noether identities are excluded from our discussion a priori. Furthermore, we shall assume that we can find an irreducible set of Noether identities

$$N_a := N_a^{A\alpha} d_\alpha G_A = 0, \tag{5.18}$$

with $N_a^{A\alpha} \in \mathrm{Loc}(\pi)$, that generate all nontrivial relations in $\mathcal{I}$. By irreducible we mean that there are no nontrivial identities among the functions $N_a^{A\alpha}$. For each Noether identity N_a we introduce a new Grassmann-even object C^*_a, called *anti-ghost*, and its formal derivatives $\partial_\alpha C^*_a \equiv C^*_{a\alpha}$:

$$\mathcal{KT} := \mathcal{K} \otimes \bigwedge \partial_\alpha C^*_a. \tag{5.19}$$

[2] Actually, for each dynamical variable, so also for each jet coordinate.

We have the obvious action of the total derivative $d_\alpha \partial_\beta C_a^* = \partial_{\alpha+\beta} C_a^*$ and extend the action of δ to $\mathcal{KT}$ by

$$\partial_\alpha C_a^* \longmapsto d_\alpha \sum_\beta N_a^{A\beta} \partial_\beta \phi_A^*, \tag{5.20}$$

which implies the grading anti-gh$(\partial_\alpha C_a^*) = 2$. Because anti-gh$(\partial_\alpha C_a^* = 2)$ the 0^{th} homology group remains unchanged when going from $\mathcal{K}$ to $\mathcal{KT}$ and, by construction of the anti-ghosts, the higher homology groups vanish:

$$H_k(\mathcal{KT}, \delta) \cong \begin{cases} \text{Loc}(\Sigma) & k = 0, \\ 0 & k \geq 1, \end{cases} \tag{5.21}$$

i.e. $(\mathcal{KT}, \delta)$ serves as a resolution for $\text{Loc}(\Sigma)$. The case of reducible Noether identities works in a very similar fashion, one simply has to introduce further anti-ghost of higher anti-gh number in order to eliminate the higher homology groups. Since the higher-spin theories we shall be considering in the following have irreducible Noether identities, we have chosen to streamline this discussion by omitting the reducible case. Our discussion on Noether identities has been substantially influenced by [Sta97].

5.4 GHOSTS

For simplicity of presentation we again restrict our discussion to the case of irreducible gauge theories. For more information [HT92] is highly recommended, as usual. The arguments we made in the finite dimensional case (Section 3.2) carry over to the infinite dimensional case, basically without modification. The gauge transformations

$$\phi^A \longmapsto \phi^A + N_a^A \epsilon^a \tag{5.22}$$

still map solutions of the equations of motion to solutions of the same and they are still integrable as long as the equations of motion hold. So, by Frobenius, they generate well-defined *gauge orbits* G_x on the stationary surface Σ. Since gauge invariant functions are, by definition, constant along the gauge orbits, they induce well-defined functions on the quotient space Σ/G, which serves as the covariant phase space in the presence of gauge invariance.

As in the finite dimensional case, we define a derivative operator γ on Σ, acting on longitudinal forms along the gauge orbits. The operator γ measures how much the longitudinal forms change along the orbits. So we again have the same description of gauge invariant functions in terms of the 0^{th} cohomology group of γ:

$$\{\text{gauge invariant functions}\} = H^0(\text{Loc}(\Sigma), \gamma). \tag{5.23}$$

The actual construction of γ is straightforward in the irreducible case. One simply adjoins Grassmann-odd elements C^a, called *ghosts*, together with their formal derivatives

$$\mathcal{BV} := \mathcal{KT} \otimes \bigwedge \partial_\alpha C^a \tag{5.24}$$

and defines the longitudinal exterior derivative $\gamma \colon \mathcal{BV} \to \mathcal{BV}$ by specifying

$$F \longmapsto \frac{\delta F}{\delta \phi^A} N_a^A C^a \qquad \text{and} \qquad \partial_\alpha C^a \longmapsto \frac{1}{2} d_\alpha \left(R_{bc}^a C^b C^c \right), \tag{5.25}$$

for any local function $F \in \mathrm{Loc}(\pi)$, where the coefficients R^a_{bc} are fixed by

$$\left[N^A_a, N^A_b\right] = R^c_{ab} N^A_c + \{\text{equations of motion}\}\,. \tag{5.26}$$

For the ghosts we introduce a new degree: pure-gh$(\partial_\alpha C^a) = 1$, called *pure ghost number*. The pure ghosts, as we shall sometimes call the ghosts introduced here, can be identified with the longitudinal one-forms dual to the vector fields defining the gauge transformations. In the reducible case, the same ideas work, one just has to introduce further, so called *ghosts of ghosts* and modify the longitudinal derivative accordingly.

Having introduced all relevant concepts, let us linger for a moment on the construction of the *Batalin-Vilkovisky complex* $\mathcal{BV}$: Let V denote the free vector space over the generating objects ϕ^*_A, C^*_a and C^a and their derivatives, and endow it with a Grassmann grading

$$V = V_0 \oplus V_1 \tag{5.27}$$

such that $C^a \in V_0$ and $\phi^*_A, C^*_a \in V_1$. Formally $\mathcal{BV}$ is now defined as

$$\mathcal{BV} := \mathrm{Loc}(\pi) \otimes \bigwedge V, \tag{5.28}$$

where $\bigwedge V$ denotes the free graded-commutative algebra over V. The derivations δ and γ as well as the gradings are specified as before. At this point we also introduce a new grading, simply called *total ghost number*, defined as gh := pure-gh − anti-gh. We summarize the field content of $\mathcal{BV}$ along with the respective gradings in the following table.

		Grassmann	pure-gh	anti-gh	gh
fields	ϕ^A	even	0	0	0
anti-fields	ϕ^*_A	odd	0	1	-1
ghosts	C^a	odd	1	0	1
anti-ghosts	C^*_a	even	0	2	-2

Table 5.1: Field content of a general irreducible gauge theory, with the corresponding gradings included.

Note that there are as many anti-fields as there are field coordinates in π and the number of ghosts and anti-ghosts is equal to the number of independent Noether identities, i.e. the construction of the ghosts and anti-ghosts depends on the particular Lagrangian, while the construction of the anti-fields does not.

5.5 THE ANTIBRACKET STRUCTURE

Having constructed the Batalin-Vilkovisky complex $\mathcal{BV}$, which serves as the arena of our continued discussion, what remains to be done is to combine the Koszul-Tate dif-

ferential with the longitudinal exterior derivative γ into the BRST operator

$$s = \delta + \gamma + \cdots, \qquad \text{such that} \qquad s^2 = 0. \tag{5.29}$$

The *main theorem* of homological perturbation theory [HT92] guarantees the existence[3] of s, and furthermore gives

$$H^g(\mathcal{BV}, s) \cong \begin{cases} 0 & g < 0, \\ H^g(\mathrm{Loc}(\Sigma), \gamma) & g \geq 0, \end{cases} \tag{5.30}$$

where g on the left-hand side denotes the total ghost number, while on the right-hand side it denotes the pure ghost number. Crucially, we get as in the finite dimensional case the isomorphism

$$H^0(\mathcal{BV}, s) \cong \{\text{gauge-invariant observables}\}. \tag{5.31}$$

In the Hamiltonian formalism we have seen that the BRST operator s was a canonical transformation in an appropriate Poisson bracket structure, i.e. $sX = [X, \Omega]$. Unfortunately, in the antifield approach there exists no suitable Poisson bracket; what does exist however, is a so called *anti-bracket*. We start by declaring that the pairs ϕ^A, ϕ^*_A and C^a, C^*_a are conjugate:

$$(\phi^A, \phi^*_B) = \delta^A{}_B, \qquad (C^a, C^*_b) = \delta^a{}_b. \tag{5.32}$$

We can extend this pairing to a map of total ghost degree $+1$

$$(-,-)\colon \mathcal{BV} \times \mathcal{BV} \longrightarrow \mathcal{BV}, \tag{5.33}$$

by defining

$$(X, Y) := \frac{\delta^{\mathrm{R}} X}{\delta \phi^A} \frac{\delta^{\mathrm{L}} Y}{\delta \phi^*_A} - \frac{\delta^{\mathrm{R}} X}{\delta \phi^*_A} \frac{\delta^{\mathrm{L}} Y}{\delta \phi^A} + \frac{\delta^{\mathrm{R}} X}{\delta C^a} \frac{\delta^{\mathrm{L}} Y}{\delta C^*_a} - \frac{\delta^{\mathrm{R}} X}{\delta C^*_a} \frac{\delta^{\mathrm{L}} Y}{\delta C^a}, \tag{5.34}$$

where the derivatives are to be interpreted as formal Euler-Lagrange derivatives. Incidentally, the same definition works for observables in their integral form, if we instead interpret the derivatives as functional derivatives. The bracket $(-,-)$ is called *anti-bracket* because of its symmetry properties and its total ghost degree of $+1$. We summarize the most important properties of the anti-bracket:

(i) it carries total ghost number $+1$:

$$\mathrm{gh}(X, Y) = \mathrm{gh}\, X + \mathrm{gh}\, Y + 1, \tag{5.35}$$

(ii) it is Grassmann-odd:

$$\varepsilon((X, Y)) = \varepsilon(X) + \varepsilon(Y) + 1, \tag{5.36}$$

(iii) it has the opposite symmetry properties as the Poisson bracket

$$(X, Y) = (-1)^{(|X|+1)(|Y|+1)+1}(Y, X), \tag{5.37}$$

(iv) it satisfies the Jacobi identity

$$(-1)^{(|X|+1)(|Z|+1)}(X, (Y, Z)) + \text{cyclic permutations} = 0, \tag{5.38}$$

(v) and $(X, -)$ acts as a graded derivation.

[3] Since we will explicitly construct the BRST operator for our particular problem, we do not actually need this general existence result for what follows.

5.6 THE MASTER EQUATION

As we already mentioned in the previous section, in the Hamiltonian formulation, the BRST operator s admitted a representation as a canonical transformation $sX = [X, \Omega]$ with respect to the Poisson structure on phase space. We shall now turn our attention to establishing an analogous result for the Lagrangian case.

If the BRST operator $s\colon \mathcal{BV} \to \mathcal{BV}$ is a canonical transformation with respect to the anti-bracket, we should be able to find some $S \in \mathcal{BV}$ such that

$$sX = (X, S), \tag{5.39}$$

for all $X \in \mathcal{BV}$. From the grading properties of the anti-bracket we get, a priori, that should S exist, it has to fulfill

$$\varepsilon(S) = 0 \qquad \text{and} \qquad \operatorname{gh} S = 0. \tag{5.40}$$

Since S has to be Grassmann-even, properties (5.36), (5.37) and (5.38) immediately give the identity

$$(S, (S, -)) = \frac{1}{2}((S, S), -), \tag{5.41}$$

and keeping the explicit form (Equation (5.34)) of the anti-bracket in mind, we see that the nilpotency of s is equivalent to the *master equation*

$$(S, S) = 0. \tag{5.42}$$

As an immediate result of standard homological perturbation theory techniques [HT92] we get the following existence and uniqueness result for the master equation in the case of irreducible gauge theories.

Theorem 5.6.1. There always exists a local functional, satisfying the appropriate gradings, that solves the master equation and in the irreducible case it is unique up to anti-canonical[4] transformations

Because of the master equation, the antibracket induces a map in cohomology, called *antibracket map*,

$$(-,-)\colon H^p(\mathcal{BV}, s) \times H^q(\mathcal{BV}, s) \longrightarrow H^{p+q+1}(\mathcal{BV}, s), \tag{5.43}$$

given by $([A], [B]) := [(A, B)]$, where $[-]$ denotes the projection onto the corresponding equivalence class in $H^\bullet(\mathcal{BV}, s)$. The antibracket map serves to illustrate a key distinction between the local formulation discussed here and a more general formulation which also allows for non-local functionals as observables. In the local case, the antibracket map contains a lot of structure while it is trivial in the non-local case, as is stated in Theorem 5.7.2. We care about this fact because the structure of the antibracket map determines all possible consistent deformations, as we will discuss in Chapter 6.

[4] Anti-canonical transformations are the obvious analogue of canonical transformations for the anti-bracket.

5.7 AUXILIARY FIELDS

We have already touched upon the subject of superfluous field content when we discussed the uniqueness of the solution of the master equation. While it is usually desired to use as few ghosts as possible, the introduction of *auxiliary fields*[5] often serves to simplify the gauge structure of the theory at hand. One would expect that the introduction of auxiliary fields does not modify the BV description in any significant way. This is in fact the case as the following theorem, as stated and proved by Henneaux [Hen90a], shows.

Theorem 5.7.1. The BRST cohomologies $H^\bullet(\mathcal{BV}, s)$ and $H^\bullet(\mathcal{BV}', s')$ of two formulations of a theory, differing only in auxiliary field content, are isomorphic.

By introducing auxiliary variables in a clever way and using the previous theorem, one can establish (see [BBH95a]) the crucial result, that the antibracket map is trivial, if one allows for non-local observables.

Theorem 5.7.2. If spacetime is globally hyperbolic and one considers as observables all functionals and not just local ones, then the antibracket map (5.43) is trivial, i.e. the antibracket of two BRST-closed functionals is BRST-exact.

[5] fields whose equations of motion admit exactly one solution

CHAPTER 6

DEFORMATIONS OF THE MASTER EQUATION

It has been realized by Barnich and Henneaux [BH93] that the problem of constructing consistent interactions between fields with gauge freedom, can be reformulated as the problem of *consistently deforming the master equation*. We shall briefly recap their arguments.

Explicit perturbative[1] construction yields the following form of the solution S of the master equation

$$S = \mathcal{L} + \phi^*_A N^A_a C^a + \cdots . \tag{6.1}$$

Notice that this solution captures all relevant information[2] about the gauge structure of the theory and the very existence of S reflects the consistency of the gauge transformations. As can already be guessed from the explicit form (6.1) of S, the original gauge invariant Lagrangian $\mathcal{L}$ can be recovered from the solution S by setting the antifields equal to zero:

$$\mathcal{L} = S\big|_{\phi^*_A = C^*_a = 0}. \tag{6.2}$$

Similarly the gauge transformations can be recovered

$$\delta_\epsilon \phi^A = \left.\frac{\delta S}{\delta \phi^*_A}\right|_{\phi^*_A = C^*_a = 0} \qquad (C^a \longrightarrow \epsilon^a). \tag{6.3}$$

Incidentally, these statements can also be formulated using the original gauge invariant action S_0 in its integral form

$$S_0 \coloneqq \int j^k \phi^\star L, \tag{6.4}$$

$$S_0[\phi^A] = S[\phi^A, \phi^*_A = C^*_a = 0], \tag{6.5}$$

$$\delta_\epsilon \phi^A = \frac{\delta S}{\delta \phi^*_A}[\phi^A, \phi^*_A = C^*_a = 0] \quad (C^a \longrightarrow \epsilon^a), \tag{6.6}$$

where we use the integral form of S and interpret the derivative in the last line as a functional derivative.

[1] in the homological sense

[2] Noether identities, closure of the gauge transformations, ...

6.1 CONSISTENT COUPLINGS AS DEFORMATIONS

Starting with a Lagrangian $\mathcal{L}$, which we shall hereafter denote by $\overset{(0)}{\mathcal{L}}$ and call *free*[3] Lagrangian, with *free* gauge symmetries

$$\phi^A \longrightarrow \phi^A + \overset{(0)}{N}{}^A_a \epsilon^a, \qquad\qquad \overset{(0)}{N}{}^A_a \frac{\delta \overset{(0)}{\mathcal{L}}}{\delta \phi^A} = 0, \tag{6.7}$$

we want to discuss the problem of introducing consistent interactions. By introducing consistent interactions, we mean that we want to modify the original Lagrangian and the original gauge transformations

$$\overset{(0)}{\mathcal{L}} \longrightarrow \mathcal{L} := \overset{(0)}{\mathcal{L}} + g\overset{(1)}{\mathcal{L}} + g^2\overset{(2)}{\mathcal{L}} + \cdots, \tag{6.8}$$

$$\overset{(0)}{N}{}^A_a \longrightarrow N^A_a := \overset{(0)}{N}{}^A_a + g\overset{(1)}{N}{}^A_a + g^2\overset{(2)}{N}{}^A_a + \cdots, \tag{6.9}$$

in such a way, that the *deformed* gauge transformations N^A_a are indeed gauge symmetries of the *deformed* Lagrangian $\mathcal{L}$

$$\big(\overset{(0)}{N}{}^A_a + g\overset{(1)}{N}{}^A_a + g^2\overset{(2)}{N}{}^A_a + \cdots\big) \frac{\delta\big(\overset{(0)}{\mathcal{L}} + g\overset{(1)}{\mathcal{L}} + g^2\overset{(2)}{\mathcal{L}} + \cdots\big)}{\delta \phi^A} = 0. \tag{6.10}$$

Even though we call deformations satisfying this condition *consistent*, it might be necessary to add further consistency requirements derived from physical and/or mathematical consideration (e.g. reducibility). Condition (6.10) is just the bare minimum required to continue using the name gauge transformations for N^A_a. On physical grounds [CP19] we shall want to disregard deformations which can be obtained (and hence also eliminated) by making field redefinitions

$$\phi \longrightarrow \phi + gF(\phi), \tag{6.11}$$

where F is analytic in g. At the Lagrangian level, this field redefinition amounts to a deformation

$$\mathcal{L} \longrightarrow \mathcal{L} + gF^A(\phi)\frac{\delta \mathcal{L}}{\delta \phi^A} + \mathcal{O}(g^2). \tag{6.12}$$

While such a deformation surely is consistent, physical observables are known to be invariant under redefinitions (6.11), and so, Lagrangians related via (6.12) are thought to describe the same physical system. We shall refer to deformations of this form as *trivial*. Similarly, deformations proportional to $\overset{(0)}{\mathcal{L}}$ can be summed as

$$\overset{(0)}{\mathcal{L}} \longrightarrow \mathcal{L} = \big(1 + k_1 g + k_2 g^2 + \cdots\big)\overset{(0)}{\mathcal{L}} \tag{6.13}$$

and hence only amount to a change in the coupling constant in front of the free action.

Definition 6.1.1. A theory defined by a Lagrangian $\mathcal{L}$ is called *rigid* if the only consistent deformations of $\mathcal{L}$ are proportional to $\mathcal{L}$, up to redefinitions of the fields.

[3] *free* should not be interpreted as placing any restriction upon the original Lagrangian whatsoever. We use the word *free* because in practice one often starts with a non-interacting Lagrangian.

6.2 COHOMOLOGICAL REFORMULATION

Simultaneously modifying $\overset{(0)}{L}$ and $\overset{(0)}{N}$ in such a way that (6.10) continues to hold to all orders in g is not entirely trivial, not least because of the complexity introduced by the apparent freedom to deform the Lagrangian and the gauge transformations independently. Recasting the problem in terms of solutions to the master equation yields a rather more economical formulation, which at the very least eliminates this apparent freedom and furthermore allows the application of powerful techniques of (co)homological deformation theory. From the previous chapter it is clear that deforming the Lagrangian and the gauge transformations subject to the condition (6.10) is equivalent to deforming the solution to the master equation of the free theory

$$\overset{(0)}{S} \longrightarrow S := \overset{(0)}{S} + g\overset{(1)}{S} + g^2\overset{(2)}{S} + \cdots , \tag{6.14}$$

such that $(S,S) = 0$ continues to hold, which guarantees our (basic) consistency requirements on the resulting Lagrangian and gauge transformations continue to hold. Splitting the master equation, order by order in the deformation parameter g, yields the system of equations:

$$\begin{aligned} 0 &= (\overset{(0)}{S},\overset{(0)}{S}) \\ 0 &= (\overset{(0)}{S},\overset{(1)}{S}) \\ 0 &= (\overset{(1)}{S},\overset{(1)}{S}) + 2(\overset{(0)}{S},\overset{(2)}{S}) \\ &\vdots \end{aligned}$$

The first equation is just the consistency requirement on the free theory. The second equation says that the first order deformation $\overset{(1)}{S}$ is a cocycle of the free BRST differential. Assume that $\overset{(1)}{S}$ is a coboundary, i.e. $\overset{(1)}{S} = (T,\overset{(0)}{S})$. Since

$$\operatorname{gh} S = 0 \quad \Rightarrow \quad \operatorname{antigh} T = \operatorname{puregh} T + 1, \tag{6.15}$$

after eliminating the antifields and antighosts, the only term in the first order deformation that does not vanish a priori is

$$(T,\overset{(0)}{S})\Big|_{\phi^*=C^*=0} = -\frac{\delta^{\mathrm{R}}T}{\delta\phi^*_A}\frac{\delta^{\mathrm{L}}\overset{(0)}{S}}{\delta\phi^A}\Bigg|_{\phi^*=C^*=0} \tag{6.16}$$

and hence at first order in g, deformations which are cocycles at first order, correspond to trivial deformations[4]

$$\overset{(0)}{\mathcal{L}} \longrightarrow \overset{(0)}{\mathcal{L}} - g\frac{\delta^{\mathrm{R}}T}{\delta\phi^*_A}\frac{\delta\overset{(0)}{\mathcal{L}}}{\delta\phi^A}. \tag{6.17}$$

Furthermore, notice that the term (6.16) vanishes for $\operatorname{antigh}(T) \neq 1$. For this reason we say that the nontrivial (local) deformations of the master equation are given by

[4] At this point in the analysis, there is no reason to expect that these deformations remain trivial at higher orders in g. They also affect the higher order structure functions in reducible gauge theories, which we shall ignore for this discussion.

$H^0(\mathcal{BV}, s)$ of the undeformed theory[5], i.e. the space of observables. Recall that elements of $\mathcal{BV}$ are only defined up to d-exact terms. In explicit calculations it is easier to consider s as acting on representatives. When doing so we have to replace $H^0(\mathcal{BV}, s)$ by the homology of s modulo d, denoted by $H^0(s|d)$. The next equation

$$s\overset{(2)}{\mathcal{S}} = -\frac{1}{2}(\overset{(1)}{\mathcal{S}}, \overset{(1)}{\mathcal{S}}), \tag{6.18}$$

says that the first order deformation $\overset{(1)}{\mathcal{S}}$ should be such that $(\overset{(1)}{\mathcal{S}}, \overset{(1)}{\mathcal{S}})$ is BRST-exact. At this point our general discussion has to stop, because the existence of such a local functional $\overset{(2)}{\mathcal{S}}$ already depends on the precise structure of the antibracket map of the theory at hand. Incidentally, the triviality of the antibracket map in the non-local case (Theorem 5.7.2) imply that there are no higher order obstructions to our deformation program if one allows non-local interactions.

[5]The differential s shall always denote the BRST operator of the undeformed theory.

CHAPTER 7

HIGHER-SPIN GAUGE THEORY

7.1 THE FRONSDAL FORMULATION

We shall consider a theory of massless particles of integer spin in the formulation originally introduced by Fronsdal [Fro78]. On a flat[1] background $\eta_{\mu\nu}$, a massless particle of spin $s \in \mathbb{N}_{\geq 4}$ is hence described by a completely symmetric field $h_{\mu_1\dots\mu_s}$ of rank s, subject to the algebraic constraint

$$0 = h''_{\mu_5\dots\mu_s} \equiv \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}h_{\mu_1\dots\mu_s}. \tag{7.1}$$

By $(-)'$ we shall always denote taking the trace of a completely symmetric field and any completely symmetric field satisfying condition (7.1) we shall call *double traceless*. The free equations of motion

$$0 = G_{\mu_1\dots\mu_s} := F_{\mu_1\dots\mu_s} - \frac{s(s-1)}{4}\eta_{(\mu_1\mu_2}F'_{\mu_3\dots\mu_s)}, \tag{7.2}$$

with

$$F_{\mu_1\dots\mu_s} := \Box h_{\mu_1\dots\mu_s} - s\partial^2_{\rho(\mu_1}h^{\rho}_{\mu_2\dots\mu_s)} + \frac{s(s-1)}{2}\partial^2_{(\mu_1\mu_2}h'_{\mu_3\dots\mu_s)}, \tag{7.3}$$

are a natural generalization of the linearized Einstein equations (i.e. the case $s = 2$). Accordingly, the tensor $G_{\mu_1\dots\mu_s}$ is called *Einstein-like tensor* and $F_{\mu_1\dots\mu_s}$ is called *Fronsdal tensor*. The curved brackets denote complete symmetrization over the enclosed indices, normalized such that $h^i_{(\mu_1\dots\mu_{s_i})} = h^i_{\mu_1\dots\mu_{s_i}}$. The correct gauge transformations to consider in this setting are

$$h_{\mu_1\dots\mu_s} \longrightarrow h_{\mu_1\dots\mu_s} + \partial_{(\mu_1}\epsilon_{\mu_2\dots\mu_s)}, \tag{7.4}$$

where the gauge parameters ϵ are completely symmetric and traceless. The free equations of motion (7.1) can also be formulated via an action principle using the action

$$S = \frac{1}{2}\int d^Dx \left(G_{\mu_1\dots\mu_s}h^{\mu_1\dots\mu_s} - \frac{s(s-1)}{4}G'_{\mu_1\dots\mu_{s-2}}h'^{\mu_1\dots\mu_{s-2}} \right), \tag{7.5}$$

where we have to impose the double-traceless constraint on the fields $h_{\mu_1\dots\mu_s}$ as well. The action S is also invariant under the gauge transformation (7.4) only if the gauge parameters are traceless.

[1] For concreteness we shall choose the signature of η to be $(-,+,\cdots,+)$, notice however, that this specific form of η does not enter into any of the following considerations.

Today, this formulation introduced by Fronsdal is called *constrained*, *metric-like* approach. Constrained, because of the trace constraints on the fields and gauge parameters, and metric-like, because $h_{\mu\nu}$ takes the role of the metric tensor in the case $s = 2$. Alternatively, another approach to higher spins can be formulated as a generalization of gravity, described in terms of vielbein and spin connection. This approach is called *frame-like* formulation, see [Bek+20] and references therein for more information. Recently [FS02], it was realized by Francia and Sagnotti that the constrained theory can be obtained by gauge fixing a particular non-local theory that does not have any trace constraints on the fields or gauge parameters. This latter, non-local theory, is sometimes referred to as the *unconstrained* approach. See [Sor05] for a pedagogical introduction to both, constrained and unconstrained approaches.

So far we have only considered a single field of a certain specified spin s. The theory we wish to consider should, in general, contain multiple fields of different spins. To this end we introduce the fields $h^i_{\mu_1\dots\mu_{s_i}}$, for $i \in \{1,\dots,n\}$, each describing a particle species of spin $s_i \in \mathbb{N}_{\geq 4}$. Of course, each of the fields is completely symmetric in its tensor indices and fulfills the double-traceless constraint (7.1). The flat-spacetime spin-s_i Einstein-like and Fronsdal tensors $G^i_{\mu_1\dots\mu_{s_i}}$ and $F^i_{\mu_1\dots\mu_{s_i}}$ are of course given by

$$G^i_{\mu_1\dots\mu_{s_i}} = F^i_{\mu_1\dots\mu_{s_i}} - \frac{s_i(s_i-1)}{4}\eta_{(\mu_1\mu_2}F'^i_{\mu_3\dots\mu_{s_i})}, \tag{7.6}$$

$$F^i_{\mu_1\dots\mu_{s_i}} = \Box h^i_{\mu_1\dots\mu_{s_i}} - s_i\partial^2_{\rho(\mu_1}h^{i\rho}_{\mu_2\dots\mu_{s_i})} + \frac{s_i(s_i-1)}{2}\partial^2_{(\mu_1\mu_2}h'^i_{\mu_3\dots\mu_{s_i})}. \tag{7.7}$$

Remark. The lowercase Latin super-/subscript $i \in \{1,\dots,n\}$ only serves to distinguish the different fields in the theory and is not part of the Einstein summation convention. There will be no tacit summations involving Latin indices. Whether, for a given field, the Latin index appears as a superscript or a subscript depends only on which is more convenient to place among the remaining indices.

The gauge transformations in this case are

$$h^i_{\mu_1\dots\mu_{s_i}} \longrightarrow h^i_{\mu_1\dots\mu_{s_i}} + \partial_{(\mu_1}\epsilon^i_{\mu_2\dots\mu_{s_i})}, \tag{7.8}$$

where each ϵ^i is completely symmetric and traceless and the free dynamics are described by the equations of motion $G^i_{\mu_1\dots\mu_{s_i}} = 0$, which can equally be formulated by the gauge invariant action

$$S = \sum_{i=1}^{n} S_i, \tag{7.9}$$

where each S_i is defined according to (7.5).

Notice that the constrained, metric-like theory is an irreducible gauge theory and hence a prime candidate to be formulated in terms of the antifield-BRST formalism, which we shall do in the following section.

7.2 BRST-FORMULATION

In this section we construct the BRST spectrum of a free theory containing multiple fields h^i of spin s_i. This can be viewed as a generalization of [BBC06], where the following construction is given for the specific case $s = 3$. Also compare with [BLS08].

For each Grassmann-even gauge parameter ϵ^i, cf. (7.8), we introduce the completely symmetric, traceless Grassmann-odd ghost $C^i_{\mu_1\dots\mu_{s_i-1}}$. Then, for each field $h^i_{\mu_1\dots\mu_{s_i}}$ we add a corresponding antifield $h_i^{*\mu_1\dots\mu_{s_i}}$ and for each ghost $C^i_{\mu_1\dots\mu_{s_i-1}}$ we add a corresponding antighost $C_i^{*\mu_1\dots\mu_{s_i-1}}$. The antifields (antighosts) satisfy the same algebraic relations as the original fields (ghosts). The BRST spectrum is summarized in Table 7.1.

		Grassmann	pure-gh	anti-gh	gh
fields	$h^i_{\mu_1\dots\mu_{s_i}}$	even	0	0	0
anti-fields	$h_i^{*\mu_1\dots\mu_{s_i}}$	odd	0	1	-1
ghosts	$C^i_{\mu_1\dots\mu_{s_i-1}}$	odd	1	0	1
anti-ghosts	$C_i^{*\mu_1\dots\mu_{s_i-1}}$	even	0	2	-2

Table 7.1: Field content of the higher-spin gauge theory under consideration, with the corresponding gradings included.

The next step then is to introduce the Koszul-Tate differential δ and the longitudinal derivative γ and combine them into the BRST differential[2] s. Recall (Chapter 5) that the Koszul-Tate differential has to generate the equations of motion and the Noether identities, while the longitudinal derivative generates the gauge transformations. This means that δ and γ have vanishing action on the fields of the theory, except for

$$\delta h_i^{*\mu_1\dots\mu_{s_i}} = G_i^{\mu_1\dots\mu_{s_i}}, \tag{7.10}$$

$$\delta C_i^{*\mu_1\dots\mu_{s_i-1}} = -s_i\left(\partial_{\mu_{s_i}} h_i^{*\mu_1\dots\mu_{s_i}} - \frac{(s_i-1)(s_i-2)}{2(D+2s_i-6)}\eta^{(\mu_1\mu_2}\eta_{\alpha\beta}\partial_{\mu_{s_i}} h_i^{*\alpha\beta|\mu_3\dots\mu_{s_i-1})\mu_{s_i}}\right), \tag{7.11}$$

$$\gamma h^i_{\mu_1\dots\mu_{s_i}} = s_i\partial_{(\mu_1} C^i_{\mu_2\dots\mu_{s_i})}. \tag{7.12}$$

In the notation N^A_a of Chapter 5, the index A corresponds to the multi-index $\mu_1\dots\mu_{s_i}$ and the index a corresponds to the multi-index $\mu_1\dots\mu_{s_i-1}$. See for example [BLS08]. At this point we want to combine γ and δ into the BRST differential s. Usually one would employ homological perturbation theory techniques (see for example[HT92]), but since

$$(\delta+\gamma)^2 = 0, \tag{7.13}$$

we can simply set $s := \delta + \gamma$. To convince yourself of (7.13) recall $\gamma G_i^{\mu_1\dots\mu_{s_i}} = 0$, because the equations of motion are gauge invariant. In order to reduce the amount of indices

[2]We use the symbol s for the BRST differential and the symbols s_i for the spin values of the fields. Hence, no confusion should be possible.

and make the notation more manageable we introduce (a kind of) generating functions for all the (anti-)fields and (anti-)ghosts of the theory. Furthermore, it turns out be beneficial to also introduce generating functions for the Fronsdal and Einstein-like tensors as well. The inspiration to study this problem using generating function comes from [FKM20], however, generating functions have been used in higher-spin gauge theory prior to this. We construct the generating functions by contracting each free spacetime index with auxiliary, spacetime independent, Grassmann-even vector variables a_i^μ:

$$G_i(x_i,a_i) \coloneqq \frac{1}{s_i!} G_i^{\mu_1\dots\mu_{s_i}}(x_i)\, a_{i\,\mu_1}\cdots a_{i\,\mu_{s_i}}, \qquad F_i(x_i,a_i) \coloneqq \frac{1}{s_i!} F_i^{\mu_1\dots\mu_{s_i}}(x_i)\, a_{i\,\mu_1}\cdots a_{i\,\mu_{s_i}},$$
$$h^i(x_i,a_i) \coloneqq \frac{1}{s_i!} h^i_{\mu_1\dots\mu_{s_i}}(x_i)\, a_i^{\mu_1}\cdots a_i^{\mu_{s_i}}, \qquad h_i^*(x_i,a_i) \coloneqq \frac{1}{s_i!} h_i^{*\mu_1\dots\mu_{s_i}}(x_i)\, a_{i\,\mu_1}\cdots a_{i\,\mu_{s_i}},$$
$$C^i(x_i,a_i) \coloneqq \frac{1}{(s_i-1)!} C^i_{\mu_1\dots\mu_{s_i-1}}(x_i)\, a_i^{\mu_1}\cdots a_i^{\mu_{s_i-1}},$$
$$C_i^*(x_i,a_i) \coloneqq \frac{1}{(s_i-1)!} C_i^{*\mu_1\dots\mu_{s_i-1}}(x_i)\, a_{i\,\mu_1}\cdots a_{i\,\mu_{s_i-1}}.$$

Notice how the symmetry in the spacetime indices is reflected in this construction. We also introduced multiple spacetime points x_i to help differentiate spacetime derivatives acting on different fields. This is only to simplify notation and has no physical significance whatsoever. The individual fields can easily be recovered from the generating functions by taking derivatives with respect to the auxiliary variables:

$$h^i_{\mu_1\dots\mu_{s_i}} = \frac{\partial}{\partial a_i^{\mu_1}}\cdots\frac{\partial}{\partial a_i^{\mu_{s_i}}} h^i. \tag{7.14}$$

This relation is the reason for the inclusion of the normalization factor in the definition of the generating functions. We shall usually suppress the explicit dependence on the spacetime points and auxiliary variables in our notation: $h^i \equiv h^i(x_i,a_i)$. Introducing the following shorthand notation for the various differential operators

$$A^i_\mu \coloneqq \frac{\partial}{\partial a_i^\mu}, \qquad z_{ij} \coloneqq A_i\cdot A_j\big|_{1\le i\le j\le n}, \qquad z_i \coloneqq z_{ii}, \tag{7.15}$$
$$P^i_\mu \coloneqq \frac{\partial}{\partial x_i^\mu}, \qquad y_{ij} \coloneqq A_i\cdot P_j\big|_{1\le i\le j\le n}, \qquad y_i \coloneqq y_{ii} \tag{7.16}$$

allows us to write the defining relations of the Koszul-Tate differential and the longitudinal derivative, Equations (7.10) to (7.12), as

$$\delta h_i^* = G_i, \tag{7.17}$$
$$\delta C_i^* = s_i\left(\frac{a^2 z_i}{2(D+2s_i-6)} - 1\right) y_i\, h_i^*, \tag{7.18}$$
$$\gamma h_i = (a_i\cdot\partial) C^i, \tag{7.19}$$

and Equations (7.6) and (7.7) as

$$G_i = \left(1-\frac{a_i^2}{4}z_i\right) F_i, \qquad F_i = \left(\Box - (a_i\cdot\partial)y_i + \frac{a_i^2}{2}(a_i\cdot\partial)^2 z_i\right) h^i. \tag{7.20}$$

When there is no risk of confusion, we shall often suppress the Latin indices. The double traceless condition of the fields and antifields takes the form

$$z^2\, h = z^2\, h^* = 0. \tag{7.21}$$

This, together with the following lemma, implies the traceless property of δC^*:

$$z\, \delta C^* = 0. \tag{7.22}$$

Lemma 7.2.1. Let ϕ be homogeneous in a of order t, i.e. of the form $\phi_{\mu_1\ldots\mu_t} a^{\mu_1}\cdots a^{\mu_t}$, where $\phi_{\mu_1\ldots\mu_{s_i}}$ is an arbitrary expression, independent of the auxiliary variables a^μ, then

$$z\,(a^2\phi) = (2D+4t)\phi + a^2 z\,\phi.$$

Proof. Simply compute

$$\begin{aligned} z\,(a^2\phi) &= \eta_{\mu\nu} A^\mu A^\nu (a^2\phi) \\ &= \eta_{\mu\nu} A^\mu \left(2\,a^\nu \phi + a^2 A^\nu \phi\right) \\ &= \eta_{\mu\nu} \left(2\eta^{\mu\nu}\phi + 2\,a^\nu A^\mu \phi + 2\,a^\mu A^\nu \phi + a^2 A^\mu A^\nu \phi\right) \\ &= 2D\,\phi + 4\,\eta_{\mu\nu} a^\mu A^\nu \phi + a^2 A^2 \phi \\ &= (2D+4t)\,\phi + a^2 z\,\phi. \end{aligned}$$

□

Having established this lemma we can compute $z\,\delta C^*$ directly from Equation (7.18) and find that the result vanishes. The Noether identities for the Einstein-like tensor

$$\partial^{\mu_{s_i}} G^i_{\mu_1\ldots\mu_{s_i}} - \frac{(s_i-1)(s_i-2)}{2(D+2s_i-6)} \eta_{(\mu_1\mu_2} \partial^{\mu_{s_i}} G'^i_{\mu_3\ldots\mu_{s_i-1})\mu_{s_i}} \equiv 0 \tag{7.23}$$

take the form

$$N_i G_i := s_i \left(\frac{a^2 z_i}{2(D+2s_i-6)} - 1\right) y_i G_i \equiv 0, \tag{7.24}$$

which is just the expression for δC_i, as expected. The double traceless condition (7.21) of the fields implies the same for the Fronsdal tensor:

$$z^2 F = 0. \tag{7.25}$$

This allows us to write the Noether identities in terms of the Fronsdal tensor:

$$yF = \frac{1}{2}(a\cdot\partial) z F, \tag{7.26}$$

as the following computation shows:

$$\begin{aligned} 0 &= \left(1 - \frac{a^2 z}{2(D+2s-6)}\right) yG \\ &= \left(1 - \frac{a^2 z}{2(D+2s-6)}\right) y \left(1 - \frac{a^2 z}{4}\right) F \\ &= \left(y - \frac{a^2 z y}{2(D+2s-6)} - \frac{y a^2 z}{4} + \frac{a^2 z y a^2 z}{8(D+2s-6)}\right) F \\ &= \left(y - \frac{a^2 z y}{2(D+2s-6)} - \frac{1}{4}\left(2(a\cdot\partial)z + a^2 y z\right) + \frac{a^2 y (2D+4s-8) z}{8(D+2s-6)}\right) F \\ &= \left(y - \frac{1}{2}(a\cdot\partial) z\right) F, \end{aligned}$$

where we used

$$ya^2zF = 2(a\cdot\partial)zF + a^2yzF \qquad \text{and} \qquad za^2zF = (2D+4s-8)zF.$$

The point of the previous two pages was to give a description of the BRST formulation of higher-spin gauge theory, entirely in terms of generating functions. For convenience and because repetition is no sin, we summarize again how the Koszul-Tate differential and longitudinal derivative act

$$\delta h^* = G, \qquad \delta C^* = N\,h^*, \qquad \gamma h = (a\cdot\partial)\,C, \tag{7.27}$$

in terms of the generating functions, and the spectrum of the theory, along with the (a priori imposed) algebraic constraints, is listed in Table 7.2.

		Grassmann	pure-gh	anti-gh	gh	algebraic constraints
fields	h^i	even	0	0	0	$z^2h = 0$
anti-fields	h^*_i	odd	0	1	-1	$z^2h^* = 0$
ghosts	C^i	odd	1	0	1	$z\,C = 0$
anti-ghosts	C^*_i	even	0	2	-2	$z\,C^* = 0$

Table 7.2: Field content of the higher-spin gauge theory in terms of the generating functions, with the corresponding gradings and algebraic constraints included.

7.3 LONGITUDINAL EXTERIOR COHOMOLOGY

So far we have been quite successful in hiding the spacetime indices of our tensor fields inside of generating functions. The reason for our success is the fact that the indices of all objects we have considered so far are completely symmetric, which plays nicely with our Grassmann-even auxiliary variables a^μ. If we would want to discuss objects whose indices are not completely symmetric, as is indeed the case in this section, we will have to come up with some new tricks.

When discussing the gauge-invariant functions, or equivalently the longitudinal exterior cohomology, there is a new object which arises quite naturally. The spin $s_i > 2$ curvature tensor K^i of a higher-spin gauge theory is defined as (see for example [BLS08])

$$K^i_{\mu_1\nu_1|\ldots|\mu_{s_i}\nu_{s_i}} := 2^{s_i}\pi^{s_i}\left(\partial^{s_i}_{\mu_1\ldots\mu_{s_i}} h^i_{\nu_1\ldots\nu_{s_i}}\right), \tag{7.28}$$

where we introduced the permutation operator

$$\pi^{s_i} := \frac{1}{2^{s_i}}\prod_{j=1}^{s_i}[e - (\mu_j\nu_j)], \tag{7.29}$$

which performs anti-symmetrization over the pairs (μ_i, ν_i), $i = 1, \ldots, s_i$. The precise definition of the curvature K is not really relevant to our computation. What is important for us however are the following relations between the Fronsdal and curvature tensors:

$$\eta^{\rho\sigma}K^i_{\rho\nu_{s_i-1}|\sigma\nu_{s_i}|\mu_1\nu_1|\ldots|\mu_{s_i-2}\nu_{s_i-2}} = 2^{s_i-2}\pi^{s_i-2}\left(\partial_{\mu_1}\cdots\partial_{\mu_{s_i-2}}F^i_{\nu_1\ldots\nu_{s_i}}\right), \tag{7.30}$$

$$\partial^{\mu_{s_i}}K^i_{\mu_1\nu_1|\ldots|\mu_{s_i}\nu_{s_i}} = 2^{s_i-1}\pi^{s_i-1}\left(\partial_{\mu_1}\cdots\partial_{\mu_{s_i-1}}F^i_{\nu_1\ldots\nu_{s_i}}\right). \tag{7.31}$$

Notice that by definition the curvature tensor is anti-symmetric in $\mu_k\nu_k$ and symmetric under the exchange of any two pairs $\mu_k\nu_k$ and $\mu_l\nu_l$. To reflect these new symmtery properties we will introduce new auxiliary, anti-symmetric, Grassmann-even variables $b_i^{\mu\nu} = -b_i^{\nu\mu}$ and define the generating functions

$$K^i := \frac{1}{s_i!}K^i_{\mu_1\nu_1|\ldots|\mu_{s_i}\nu_{s_i}}b_i^{\mu_1\nu_1}\cdots b_i^{\mu_{s_i}\nu_{s_i}}, \tag{7.32}$$

$$d^k h^i_{\mu_1\ldots\mu_{s_i-k}} := \frac{1}{k!}\partial_{\rho_1}\cdots\partial_{\rho_k}h^i_{\nu_1\ldots\nu_k\mu_1\ldots\mu_{s_i-k}}b_i^{\rho_1\nu_1}\cdots b_i^{\rho_k\nu_k}, \tag{7.33}$$

$$d^k C^i_{\mu_1\ldots\mu_{s_i-k-1}} := \frac{1}{k!}\partial_{\rho_1}\cdots\partial_{\rho_k}C^i_{\nu_1\ldots\nu_k\mu_1\ldots\mu_{s_i-k-1}}b_i^{\rho_1\nu_1}\cdots b_i^{\rho_k\nu_k}, \tag{7.34}$$

$$d^k h^i := \frac{1}{(s_i-k)!}d^k h^i_{\mu_1\ldots\mu_{s_i-k}}a_i^{\mu_1}\cdots a_i^{\mu_{s_i-k}}, \tag{7.35}$$

$$d^k C^i := \frac{1}{(s_i-k-1)!}d^k C^i_{\mu_1\ldots\mu_{s_i-k-1}}a_i^{\mu_1}\cdots a_i^{\mu_{s_i-k-1}}, \tag{7.36}$$

$$d^k F^i := \frac{1}{k!(s_i-k)!}\partial_{\mu_1}\cdots\partial_{\mu_k}F^i_{\nu_1\ldots\nu_{s_i}}b_i^{\mu_1\nu_1}\cdots b_i^{\mu_k\nu_k}a_i^{\nu_{k+1}}\cdots a_i^{\nu_{s_i}}, \tag{7.37}$$

for $1 \leq k \leq s_i - 1$. While explicitly defining the only partially contracted relations (7.33) and (7.34) is not strictly necessary it is a convenient way to introduce anti-symmetrization over pairs of indices, and hence makes some of the following arguments more transparent. Notice that they are completely symmetric in their space-time

indices. Introducing the corresponding differential operators, cf. Equations (7.15) and (7.16),

$$B^i_{\mu\nu} := \frac{1}{2}\frac{\partial}{\partial b^{\mu\nu}_i}, \qquad \check{z}_i := \frac{1}{2}a_i^{\nu_1}a_i^{\nu_2}\eta^{\mu_1\mu_2}B^i_{\mu_1\nu_1}B^i_{\mu_2\nu_2}, \qquad \check{y}_i := a_i^{\nu}\eta^{\mu\rho}P^i_{\rho}B^i_{\mu\nu}, \tag{7.38}$$

allows us to recover the original tensors and write Equations (7.30) and (7.31) in the convenient form

$$\check{z}\,K = 2^{s-2}d^{s-2}F, \qquad \check{y}\,K = 2^{s-1}d^{s-1}F. \tag{7.39}$$

In [BB05], Bekaert and Boulanger prove the following result, for the case of a single constrained[3] higher-spin gauge field. However, their result immediately generalizes to a theory of multiple non-interacting higher-spin gauge fields.

Theorem 7.3.1. The cohomology of the longitudinal exterior derivative γ is the superalgebra freely generated by[4]

(i) the spacetime coordinates x^μ,

(ii) the Fronsdal tensor and its partial derivatives,

(iii) the antifields and antighosts and their partial derivatives,

(iv) the traceless components of the partial derivatives of the curvature tensor, and

(v) the traceless parts $\widehat{d^kC^i}$ of the exterior[5] derivatives of the ghosts.

Explicitly, for f and g local functions, and fixed $i \in \{1, \dots, n\}$

$$\gamma f\big(x, [h^i], [C^i], [h^*_i], [C^*_i]\big) = 0 \quad \Leftrightarrow \quad f = g\big(x, \widehat{[K^i]}, [F^i], \widehat{d^kC^i}, [h^*_i], [C^*_i]\big) + \gamma(\cdots),$$

where

$$g\big(x, \widehat{[K^i]}, [F^i], \widehat{d^kC^i}, [h^*_i], [C^*_i]\big) = \gamma(\cdots) \quad \Leftrightarrow \quad g = 0.$$

Corollary 7.3.2. As a corollary of this result we get that the most general local, gauge-invariant function must depend on h^i only through the Fronsdal tensor F^i and the traceless components of (the partial derivatives of) the curvature K^i. Therefore, on-shell it is a function of the traceless components of the Weyl-like tensor $\mathcal{W}$ only:

$$\delta_\epsilon g([h^i]) = 0, \quad \forall \epsilon \qquad \Leftrightarrow \qquad g = g(\widehat{[K]}, [F]) \qquad \Leftrightarrow \qquad g \approx f(\widehat{[\mathcal{W}^i]}),$$

where $f(\widehat{[K^i]}) := g(\widehat{[K^i]}, [F^i] = 0)$. The *Weyl-like tensor* $\mathcal{W}^i$ is defined to be the traceless part of the curvature tensor K^i. In [BB03] it is argued, cf. Equation (7.30), that the trace of the curvature tensor vanishes on-shell, hence we have $K^i \approx \mathcal{W}^i$.

Let $\{\omega^\alpha\}$ be a basis of the space of polynomials in the ghosts and their exterior derivatives. We denote this space by $\mathcal{P}_{\text{gh}}$. Consistent with our convention in Section 2.1

[3] They also proved the analogous result for the unconstrained theory.

[4] Note that we only use the notation introduced in Equation (7.34) for convenience. The theorem is formulated entirely without generating functions.

[5] This is a kind of generalization of the ordinary exterior derivative in the sense of [DH99] and [DH02]; see also [BB04] for a discussion that is directly applicable to [BB05].

we use Greek letters from the beginning of the alphabet to indicate spacetime multi-indices. Since there are a finite number of ghosts and a finite number of exterior derivatives and since these variables are anti-commuting, the space generated by them is finite-dimensional. We use this to notationally separate the ghosts from the rest of the variables. In this construction we mirror the analogous construction in [BBC06]. If a is a γ-closed, local form we can write

$$\gamma a = 0 \qquad \Rightarrow \qquad a = \kappa_\alpha\big(\widehat{[K^i]}, [F^i], [h_i^*], [C_i^*]\big)\,\omega^\alpha + \gamma b. \tag{7.40}$$

Because a only contains a finite number of ghosts, if $\mathrm{gh}\, a$ is fixed and finite, it can only contain a finite number of antifields and antighosts. Moreover, since we require a to be a local form, it can only contain a finite number of derivatives. This means that also the coefficients κ_α are polynomials. Such a polynomial $\kappa_\alpha\big(\widehat{[K^i]}, [F^i], [h_i^*], [C_i^*]\big)$ we shall call *invariant polynomial*, and we denote the space of invariant polynomials by $\mathcal{P}_{\mathrm{loc}}$. The relevance of $\mathcal{P}_{\mathrm{loc}}$ will become clear only later, when we establish that the cohomology of δ modulo d in the space of invariant polynomials is crucially involved in describing the possible deformations of the master equation.

Definition 7.3.3 (differential d')**.** We define d' be specifying how it acts on the fundamental fields in the theory. On the fields, the antifields, the antighosts and all their derivatives d' acts just like d, but its action on the ghosts and their exterior derivatives is given by[6]

$$d' C^i_{\mu_1\ldots\mu_{s_i-1}} := \frac{2}{s_i} dx^\nu \sum_{r=1}^{s_i-1} B_{\nu\mu_r} \widehat{d^1 C^i}_{\mu_1\ldots\widehat{\mu_r}\ldots\mu_{s_i-1}}, \tag{7.41}$$

$$d' \widehat{d^k C^i}_{\mu_1\ldots\mu_{s_i-k-1}} := \frac{2}{s_i} dx^\nu \sum_{r=1}^{s_i-1} B_{\nu\mu_r} \widehat{d^{k+1} C^i}_{\mu_1\ldots\widehat{\mu_r}\ldots\mu_{s_i-1}}, \quad \text{for } 1 \le k \le s_i - 2, \tag{7.42}$$

$$d'(\partial_{\nu_1}\cdots\partial_{\nu_r} C^i_{\mu_1\ldots\mu_{s_i-1}}) := 0, \qquad \text{for } r \ge s_i - 1. \tag{7.43}$$

Definition 7.3.4 (d'-degree)**.** For terms of the form $\kappa_I \omega^I$, with κ_I an invariant polynomial, define its d'-degree to be the number of $d^k C^i$'s contained in the term, weighted by the factor k, i.e. the number of $d^1 C^i$'s plus two times the number of $d^2 C^i$'s and so on. It is bounded because there is a finite number of $d^k C^i$'s and they are anti-commuting.

Notice that d' splits as the sum of an operator d'_1 which raises the d'-degree by one unit, and an operator d'_0 which leaves it unchanged. The operator d'_0 gives 0 when acting on the ghosts and agrees with d on the rest of the spectrum. The operator d'_1 reproduces d' when acting on the ghosts and gives 0 otherwise. The relations

$$\partial_{\mu_1} C^i_{\mu_2\ldots\mu_{s_i}} = \frac{1}{s_i}\gamma\, h^i_{\mu_1\ldots\mu_{s_i}} + \frac{2}{s_i}\sum_{r=2}^{s_i} B_{\mu_1\mu_r} d^1 C^i_{\mu_2\ldots\widehat{\mu_r}\ldots\mu_{s_i}}, \tag{7.44}$$

$$\partial_{\mu_1} d^k C^i_{\mu_2\ldots\mu_{s_i-k}} = \frac{1}{s_i-k}\gamma\, d^k h^i_{\mu_1\ldots\mu_{s_i-k}} + \frac{2}{s_i-k}\sum_{r=2}^{s_i-k} B_{\mu_1\mu_r} d^{k+1} C^i_{\mu_2\ldots\widehat{\mu_r}\ldots\mu_{s_i-k}}, \tag{7.45}$$

$$\partial_\mu d^{s_i-1} C^i = \gamma\, d^{s_i-1} h^i_\mu, \tag{7.46}$$

[6] Again, we use the notation of generating functions, but this action could just as well be interpreted in terms of the ungarnished fields. The diacritic $\widehat{\ }$ indicates the traceless part of the field, as always.

where $1 \leq k \leq s_i - 2$, imply that d' agrees with d up to γ-exact terms. These relations are straightforward to derive, as the following computations shows, where $0 \leq k \leq s_i - 1$:

$$\begin{aligned}
\gamma\, d^k h^i_{\mu_1\dots\mu_{s_i-k}} &= \frac{1}{k!}\partial_{\rho_1}\cdots\partial_{\rho_k}\gamma\, h^i_{\nu_1\dots\nu_k\mu_1\dots\mu_{s_i-k}} b_i^{\rho_1\nu_1}\cdots b_i^{\rho_k\nu_k} \\
&= \frac{s_i}{k!}\partial_{\rho_1}\cdots\partial_{\rho_k} C^i_{(\nu_1\dots\nu_k\mu_1\dots\mu_{s_i-k-1},\mu_{s_i-k})} b_i^{\rho_1\nu_1}\cdots b_i^{\rho_k\nu_k} \\
&= \frac{s_i}{k!}\frac{s_i-k}{s_i}\partial_{\rho_1}\cdots\partial_{\rho_k} C^i_{\nu_1\dots\nu_k(\mu_1\dots\mu_{s_i-k-1},\mu_{s_i-k})} b_i^{\rho_1\nu_1}\cdots b_i^{\rho_k\nu_k} \\
&= (s_i-k)\partial_{(\mu_1} d^k C^i_{\mu_2\dots\mu_{s_i-k})} \\
&= \sum_{r=1}^{s_i-k}\partial_{\mu_r} d^k C^i_{\mu_1\dots\widehat{\mu}_r\dots\mu_{s_i-k}} \\
&= \partial_{\mu_1} d^k C^i_{\mu_2\dots\mu_{s_i-k}} + \sum_{r=2}^{s_i-k}\partial_{\mu_r} d^k C^i_{\mu_1\dots\widehat{\mu}_r\dots\mu_{s_i-k}} \\
&= \partial_{\mu_1} d^k C^i_{\mu_2\dots\mu_{s_i-k}} + \sum_{r=2}^{s_i-k}\left(\partial_{\mu_1} d^k C^i_{\mu_2\dots\mu_{s_i-k}} + 2\, d^k C^i_{\mu_2\dots\widehat{\mu}_r\dots\mu_{s_i-k}[\mu_1,\mu_r]}\right) \\
&= (s_i-k)\partial_{\mu_1} d^k C^i_{\mu_2\dots\mu_{s_i-k}} - 2\sum_{r=2}^{s_i-k} B_{\mu_1\mu_r} d^{k+1} C^i_{\mu_2\dots\widehat{\mu}_r\dots\mu_{s_i-k}}
\end{aligned}$$

Proposition 7.3.5 (Cohomology of d)**.** In form degree less than D and in antifield number strictly greater than 0, the cohomology of d is trivial in the space of invariant polynomials.

This proposition is stated and proved in [Bou+01] for the case $s = 2$, however, the argument presented there goes through for any $s > 1$, basically unchanged.

Theorem 7.3.6. For antigh $a > 0$, the equation $\gamma a + db = 0$ is equivalent, up to trivial redefinitions, to $\gamma a = 0$. By this we mean that we can always add d-exact terms to a such that $\gamma(a + dc) = 0$.

Proof. Following [Bou+01] and [BBC06] we consider the descent associated with the equation $\gamma a + db = 0$. The nilpotency of the longitudinal derivative $\gamma^2 = 0$ implies that db is γ-closed and since $\gamma d + d\gamma = 0$ we have that γb is d-closed. Since the cohomology of d is trivial in the space of invariant polynomials at positive antifield number we get the equation $\gamma b + dc = 0$, where c is some invariant polynomial. Continuing this line of argument we get a so called *descent*:

$$\begin{aligned}
\gamma a + db &= 0 \\
\gamma b + dc &= 0 \\
\gamma c + de &= 0 \\
&\vdots \\
\gamma m + dn &= 0 \\
\gamma n &= 0
\end{aligned}$$

Notice that each successive equation has one less unit of form-degree. The descent ends with $\gamma n = 0$, either because n is a zero form or because one stops earlier with a

γ closed term. A γ closed term has to be of the form $n = \kappa_\alpha \omega^\alpha + \gamma n'$. Inserting this expression into the descend gives

$$\begin{aligned} 0 &= d(\kappa_\alpha)\omega^\alpha \pm \kappa_\alpha d\omega^\alpha + \gamma(m + dn') \\ &= d(\kappa_\alpha)\omega^\alpha \pm \kappa_\alpha d'\omega^\alpha + \gamma m', \end{aligned}$$

where we absorbed dn' and possible γ-exact terms, arising due to the replacement of d with d', into a redefinition of m. Almost by definition we have $d'\omega^\alpha = A^\alpha_\beta \omega^\beta$, for some constant matrix A^α_β that involves only the dx^μ. Inserting this into the previous relation we get

$$\left(d(\kappa_\alpha) \pm \kappa_\beta A^\beta_\alpha\right) \omega^\alpha + \gamma m' = 0. \tag{7.47}$$

Since a term of the form $\lambda_\alpha \omega^\alpha$ (with λ invariant) is γ-exact if and only if it vanishes, this result implies

$$d(\kappa_\alpha)\omega^\alpha \pm \kappa_\alpha d'\omega^\alpha = 0. \tag{7.48}$$

Expanding according to d'-degree we get at lowest order $d\kappa_{\alpha_0} = 0$, where α_0 labels the basis element that contains no derivatives of the ghost. According to Proposition 7.3.5, this implies $\kappa_{\alpha_0} = d\lambda_{\alpha_0}$, for some invariant polynomial λ_{α_0}. Hence we can write

$$\kappa_{\alpha_0}\omega^{\alpha_0} = d(\lambda_{\alpha_0}\omega^{\alpha_0}) \mp \lambda_{\alpha_0} d'\omega^{\alpha_0} + \gamma\text{-exact terms}. \tag{7.49}$$

Notice that the term $\lambda_{\alpha_0} d'\omega^{\alpha_0}$ has d'-degree equal to 1. This means that by adding d-exact terms and γ-exact terms to the last term in the descent $n = \kappa_\alpha \omega^\alpha + \gamma(\cdots)$, we can remove any terms of d'-degree 0 from $\kappa_\alpha \omega^\alpha$. Continuing this argument we can remove any terms of d'-degree 1, d'-degree 2 and so on, until we get $n = \gamma(\cdots) + d(\cdots)$. This means the previous line in the descent reduces to $\gamma m = 0$, where we absorbed possible d-exact terms into m. One now repeats the whole argument until one gets $\gamma a = 0$. □

7.4 INVARIANT COHOMOLOGY OF δ MODULO d

Recall from Section 6.2, that nontrivial deformations of the master equation correspond, up to first order in the deformation parameter, to elements of the 0^{th}-cohomology group of s modulo d of the original theory. The previous theorem allows us to give a more manageable description in terms of the invariant cohomology of δ modulo d, denoted $H^{\text{inv}}(\delta|d)$. A cocycle of $H(s|d)$ is defined by

$$sa + db = 0. \tag{7.50}$$

Expanding a and b according to antifield number:

$$a = a_0 + a_1 + \cdots + a_k, \tag{7.51}$$
$$b = b_0 + b_1 + \cdots + b_l, \tag{7.52}$$

where the expansion has to stop[7] at some finite antifield number, and writing s as the sum of δ and γ allows us to write Equation (7.50) as the system of equations:

$$\begin{aligned}
\delta a_1 + \gamma a_0 + d b_0 &= 0 \\
\delta a_2 + \gamma a_1 + d b_1 &= 0 \\
&\vdots \\
\delta a_k + \gamma a_{k-1} + d b_{k-1} &= 0 \\
&\vdots
\end{aligned}$$

We may assume $l = k - 1$ without loss of generality, because if $l > k - 1$ the rest of the descend is of the form $db_i = 0$ which implies, using Proposition 7.3.5 and $i > k \geq 0$, that the additional terms $b_{i \geq k}$ can simply be absorbed into lower order terms. Furthermore, using the result from Theorem 7.3.6 we can replace the last equation $\gamma a_k + d b_k = 0$ with the equivalent equation $\gamma a_k = 0$. Hence we may instead focus our attention on the following system of equations:

$$\begin{aligned}
\delta a_1 + \gamma a_0 + d b_0 &= 0 \\
\delta a_2 + \gamma a_1 + d b_1 &= 0 \\
&\vdots \\
\delta a_k + \gamma a_{k-1} + d b_{k-1} &= 0 \\
\gamma a_k &= 0
\end{aligned}$$

We continue in a fashion similar to the proof of Theorem 7.3.6. The last equation can of course be solved to yield $a_k = \kappa_\alpha \omega^\alpha + \gamma(\cdots)$, and applying γ to the penultimate equation gives $d\gamma b_{k-1} = 0$. The triviality of d in positive antifield number and again Theorem 7.3.6 allow us to write $b_{k-1} = \lambda_\alpha \omega^\alpha + \gamma(\cdots)$. Notice that as previously stated, κ_α and λ_α can be chosen to be invariant polynomials. Inserting the explicit forms of a_k and b_{k-1} into the penultimate equation and replacing d with d' yields

$$\gamma(\cdots) = \delta(\kappa_\alpha \omega^\alpha) + d'(\lambda_\alpha \omega^\alpha) = \delta(\kappa_\alpha)\omega^\alpha + d'(\lambda_\alpha \omega^\alpha). \tag{7.53}$$

[7] this is shown in [BBH95b]

As in the proof of Theorem 7.3.6, using $d'\omega^\alpha = A^\alpha_\beta \omega^\beta$ we see that the expression in Equation (7.53) can only be γ-exact if it vanishes. Hence, this equation implies

$$\delta(\kappa_\alpha)\omega^\alpha + d'(\lambda_\alpha \omega^\alpha) = 0. \tag{7.54}$$

Again, after expanding according to d' degree, the term of degree zero reads

$$(\delta\kappa_{\alpha_0} + d'_0\lambda_{\alpha_0})\omega^{\alpha_0} = 0, \tag{7.55}$$

which implies that the coefficient of ω^{α_0} has to vanish. Since the action of d'_0 on λ_{α_0} is the same as the action of d we get

$$\delta\kappa_{\alpha_0} + d\lambda_{\alpha_0} = 0. \tag{7.56}$$

Assume for the moment $H^{\text{inv}}_k(\delta|d) = 0$. This would guarantee the existence of invariant polynomials μ_{α_0} and ν_{α_0} such that

$$\kappa_{\alpha_0} = \delta\mu_{\alpha_0} + d\nu_{\alpha_0}. \tag{7.57}$$

Denoting the d'-degree 0 component of a_k by a^0_k, we thus get

$$\begin{aligned} a^0_k &= (\delta\mu_{\alpha_0} + d'_0\nu_{\alpha_0})\omega^{\alpha_0} \\ &= s(\mu_{\alpha_0}\omega^{\alpha_0}) \pm d(\nu_{\alpha_0}\omega^{\alpha_0}) \mp \nu_{\alpha_0} d\omega^{\alpha_0} \\ &= s(\mu_{\alpha_0}\omega^{\alpha_0}) \pm d(\nu_{\alpha_0}\omega^{\alpha_0}) \mp \nu_{\alpha_0}(A^{\alpha_0}_{\beta_1}\omega^{\beta_1} + s\theta^{\alpha_0}), \end{aligned}$$

where θ^{α_0} is also a basis element of $\mathcal{P}_{\text{gh}}$ and we replaced γ and δ by s wherever possible. Notice that the term $\nu_{\alpha_0}A^{\alpha_0}_{\beta_1}\omega^{\beta_1}$ has d'-degree 1 and $\nu_{\alpha_0}s\theta^{\alpha_0}$ differs from the s-exact term $s(\pm\nu_{\alpha_0}\omega^{\alpha_0})$ only by the term $\pm(\delta\nu_{\alpha_0})\theta^{\alpha_0}$, which is of lower antifield number. This means that through trivial redefinitions we can assume that a^0_k vanishes. This now implies $d\lambda_{\alpha_0} = 0$, hence it is d-exact, hence we can also set it to zero using trivial redefinitions. Using the same procedure we can then remove the terms of d'-degree 1 and so on, successively removing terms of increasing d'-degree until one runs out of higher degree terms, since the d'-degree is finite, and has thus completely redefined away a_k and b_{k-1}.

If $H^{\text{inv}}_{k-1}(\delta|d)$ also vanishes, one can repeat this process to remove a_{k-1} and b_{k-2} and so on. In a sense $H^{\text{inv}}_\bullet(\delta|d)$ indicates if there exist any obstructions to removing terms of higher antifield number in Equations (7.51) and (7.52). For the rest of this work we shall therefore focus on computing $H^{\text{inv}}_\bullet(\delta|d)$. Our focus shall lie in particular on proving that $H^{\text{inv}}_k(\delta|d)$ vanishes for $k \geq 3$.

Recall that an invariant polynomial $\kappa_\alpha \in \mathcal{P}_{\text{loc}}$ is a polynomial in the derivatives of the antifields, antighosts and the Fronsdal tensor, and the traceless parts of the derivatives of the curvature tensor

$$\kappa^\alpha = \kappa^\alpha\left([h^*_i], [C^*_i], [F^i], \widehat{[K^i]}\right), \tag{7.58}$$

where α is a spacetime multi-index, which eventually gets contracted with elements of $\mathcal{P}_{\text{gh}}$ to produce a local form $\kappa^\alpha\omega_\alpha$. Note that the summation, in general, goes over all possible multi-indices, and so in a sense there appear several different invariant polynomials in the previous summation. When we discuss a fixed invariant polynomial κ^α, the multi-index is fixed. We want to compute the invariant cohomology of δ modulo d

in top form-degree: $H_k^{D,\mathrm{inv}}(\delta|d)$, i.e. the cohomology of δ modulo d in the space of invariant polynomials. We write an invariant polynomial κ^α with fixed anti-field number k as

$$\kappa^\alpha = \sum_{\substack{l,m \\ l+2m=k}} X^\alpha_{l,m}\Big([F],\widehat{[K]},A,P\Big)\Phi^{(l,m)}\Big|_{\substack{i_l=i \\ a_i=0 \\ x_i=x}}, \tag{7.59}$$

where $\Phi^{(l,m)} = h^*_{i_1}\cdots h^*_{i_l}C^*_{i_{l+1}}\cdots C^*_{i_{l+m}}$ is the product of l antifields and m antighosts, with $l,m\in\mathbb{N}$ such that $l+2m=k$. The operator $X^\alpha_{l,m}$ is a polynomial in the operators A and P and the functions $[F]$ and $\widehat{[K]}$. For notational simplicity we shall usually suppress the multi-index α from now on.

Right from the start we may impose several restrictions on the operators X, without losing any generality. Consider for instance a term of the form $X\,C^*_{i_1}C^*_{i_2}$. We can give a more explicit form to the operator X as

$$X[i_1,i_2] = \sum X_{\mu_1\ldots\mu_{r_1+t_1}\nu_1\ldots\nu_{r_2+t_2}}A^{\mu_1}_{i_1}\cdots A^{\mu_{r_1}}_{i_1}P^{\mu_{r_1+1}}_{i_1}\cdots P^{\mu_{r_1+t_1}}_{i_1}A^{\nu_1}_{i_2}\cdots A^{\nu_{r_2}}_{i_2}P^{\nu_{r_2+1}}_{i_2}\cdots P^{\nu_{r_2+t_2}}_{i_2},$$

where the sum goes over the numbers t_1 and t_2 and for all indices of the antighosts to be contracted we must have

$$r_1 = s_{i_1}-1 \qquad\qquad \text{and} \qquad\qquad r_2 = s_{i_2}-1, \tag{7.60}$$

and the coefficients $X_{\mu_1\ldots\nu_{r_2+t_2}}$ are functions of $[F]$ and $\widehat{[K]}$. Because in the end, cf. Equation (7.59), we identify $i_1=i_2=i$ and $x_i=x$, and the antighosts are Grassmann-even objects the operator X should reflect this symmetry:

$$X[i_1,i_2] = X[i_2,i_1]. \tag{7.61}$$

We say that X is symmetric in i_1 and i_2. Had we instead considered a term $Xh^*_{i_1}h^*_{i_2}$, we would instead, using the same arguments, get that $X[i_1,i_2]=-X[i_2,i_1]$ and call X anti-symmetric in i_1 and i_2.

We start our computation of the cohomology groups in top form degree by specifying the δ-cocycles at a fixed anti-ghost number k, which are (in dual notation) defined by the condition

$$\partial_\mu(\cdots)^\mu = \delta\kappa = \sum X_{l,m}\delta\Phi^{(l,m)}. \tag{7.62}$$

The term $h^*_{i_1}\cdots h^*_{i_m}$ is completely anti-symmetric in $i_1,\ldots,i_m$, so by the same arguments as before we may agree to only apply operators X that are completely anti-symmetric in these indices and we can write

$$\begin{aligned} X\delta(h^*_{i_1}\cdots h^*_{i_m}) &= X\sum_{k=1}^{m}(-1)^{m-k}h^*_{i_1}\cdots h^*_{i_{k-1}}G_{i_k}h^*_{i_{k+1}}\cdots h^*_{i_m} \\ &= mX\,h^*_{i_1}\cdots h^*_{i_{m-1}}G_{i_m}, \end{aligned}$$

or in a more compact form

$$\delta(h^*_{i_1}\cdots h^*_{i_m}) \sim m\,h^*_{i_1}\cdots h^*_{i_{m-1}}G_{i_m},$$

where the similarity symbol indicates that we used that the expression will be acted upon by a completely anti-symmetric operator in the relevant indices. The same reasoning in the symmetric case is applied to yield

$$\begin{aligned}\delta(C^*_{i_1}\cdots C^*_{i_m}) &= \sum_{k=1}^{m} C^*_{i_1}\cdots C^*_{i_{k-1}}\delta C^*_{i_k} C^*_{i_{k+1}}\cdots C^*_{i_m}\\ &\sim m\, C^*_{i_1}\cdots C^*_{i_{m-1}}\delta C^*_{i_m},\end{aligned}$$

which implies

$$\begin{aligned}\delta(C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{r+t}}) \sim (-1)^t r\, C^*_{i_1}\cdots C^*_{i_{r-1}}\delta C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{r+t}}\\ + t\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{r+t-1}}G_{i_{r+t}}\end{aligned}$$

In even antifield number $k = 2m$, the possible combinations of antifields are

$$C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r}}, \quad \text{for} \quad r = 0,\ldots,m, \tag{7.63}$$

which means that the image of δ at that antifield number is given by terms of the form

$$\begin{aligned}\delta(C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r}}) \sim 2(m-r)\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r-1}}G_{i_{2m-r}}\\ + r\, C^*_{i_1}\cdots C^*_{i_{r-1}}\delta C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r}},\end{aligned}$$

and Equation (7.62) takes the explicit form

$$\begin{aligned}\partial_\mu(\cdots)^\mu = \sum_{r=0}^{m} X_r\Big(&2(m-r)\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r-1}}G_{i_{2m-r}}\\ &+r\, C^*_{i_1}\cdots C^*_{i_{r-1}}\delta C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r}}\Big) \qquad (7.64)\\ = \delta\Big(\sum_{r=0}^{m} X_r\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r}}\Big), &\qquad (7.65)\end{aligned}$$

with X_r symmetric in $i_1,\ldots,i_r$ and anti-symmetric in $i_{r+1}\ldots i_{2m-r}$. We can separate the terms by antighost content and obtain the system of equations

$$\begin{aligned}&X_r\, 2(m-r)\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+1}}\cdots h^*_{i_{2m-r-1}}G_{i_{2m-r}}\\ &+ X_{r+1}(r+1)\, C^*_{i_1}\cdots C^*_{i_r}\delta C^*_{i_{r+1}}h^*_{i_{r+2}}\cdots h^*_{i_{2m-r-1}} = \partial_\mu(\cdots)^\mu,\end{aligned}$$

to be solved for X_r, for $r = 0,\ldots,m-1$. We absorb the numerical prefactors into the operators X_r and are left with the system

$$(X_r\, h^*_{i_{r+1}}G_{i_{2m-r}} + X_{r+1}\,\delta C^*_{i_{r+1}})\, C^*_{i_1}\cdots C^*_{i_r}h^*_{i_{r+2}}\cdots h^*_{i_{2m-r-1}} = \partial_\mu(\cdots)^\mu, \tag{7.66}$$

where the operators X_r and X_{r+1} act on everything to the right, not just on what is inside of the brackets. A continuing pattern of our analysis so far, has been to absorb most of the complexity into the operators X_r, simplifying the expressions we are manipulating, and leaving only the most relevant properties explicit at each step of the discussion. Considering what we started out with, Equation (7.66) is already quite manageable, however, the explicit form of δC^*_i makes it still somewhat unwieldy. We shall use the following lemma to again absorb some of this complexity into the operators X_r.

Lemma 7.4.1. Let ϕ be homogeneous in a of order t, as in Lemma 7.2.1, and $\epsilon_{\mu_1\dots\mu_r}$ be completely symmetric and traceless, then

$$\epsilon_{\mu_1\dots\mu_r} A^{\mu_1}\cdots A^{\mu_r}(a^2\phi) = \epsilon_{\mu_1\dots\mu_r}\left(2r\, a^{\mu_1}A^{\mu_2}\cdots A^{\mu_r} + a^2 A^{\mu_1}\cdots A^{\mu_r}\right)\phi. \tag{7.67}$$

Proof. Using induction on r

$$\begin{aligned}
\epsilon_{\mu_1\dots\mu_r} A^{\mu_1}\cdots A^{\mu_r}(a^2\phi) &= A^{\mu_r}\epsilon_{\mu_r\mu_1\dots\mu_{r-1}} A^{\mu_1}\cdots A^{\mu_{r-1}}(a^2\phi)\\
&= \epsilon_{\mu_1\dots\mu_r} A^{\mu_r}\Big(2(r-1)a^{\mu_1}A^{\mu_2}\cdots A^{\mu_{r-1}}\phi\\
&\qquad + a^2 A^{\mu_1}\cdots A^{\mu_{r-1}}\phi\Big)\\
&= \epsilon_{\mu_1\dots\mu_r}\Big(2(r-1)\eta^{\mu_1\mu_r}A^{\mu_2}\cdots A^{\mu_{r-1}}\phi\\
&\qquad + 2(r-1)a^{\mu_1}A^{\mu_2}\cdots A^{\mu_r}\phi\\
&\qquad + 2a^{\mu_r}A^{\mu_1}\cdots A^{\mu_{r-1}}\phi + a^2 A^{\mu_1}\cdots A^{\mu_r}\phi\Big)\\
&= \epsilon_{\mu_1\dots\mu_r}\left(2r\, a^{\mu_1}A^{\mu_2}\cdots A^{\mu_r} + a^2 A^{\mu_1}\cdots A^{\mu_r}\right)\phi. \qquad \square
\end{aligned}$$

In particular, Equation (7.67) implies that

$$\epsilon_{\mu_1\dots\mu_r} A^{\mu_1}\cdots A^{\mu_r}(a^2 z y\, h^*) = 0, \quad \text{for} \quad r > s-2. \tag{7.68}$$

So as long as we agree to always fully contract the antighosts C^* with terms that are completely symmetric and traceless, i.e. restrict ourselves to operators whose terms acting on the antighosts C^* are completely symmetric and traceless, we can set

$$\delta C_i^* := -y_i h_i^*. \tag{7.69}$$

The condition $r > s_i - 2$ is satisfied, because we consider only terms in which all $s_i - 1$ spacetime indices appearing in δC_i^* are fully contracted. This means we can give the system (7.66) the somewhat simpler form

$$(X_r\, G_{i_{2m-r}} - X_{r+1} y_{i_{r+1}})\, C_{i_1}^*\cdots C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m-r-1}}^* = \partial_\mu(\cdots)^\mu. \tag{7.70}$$

So far we have only considered the case of even antifield number. In odd antifield number $k = 2m+1$, the possible combinations of antifields are

$$C_{i_1}^*\cdots C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m+1-r}}^*, \quad \text{for} \quad r = 0,\dots,m, \tag{7.71}$$

which means that the image of δ at that antifield number is given by terms of the form

$$\begin{aligned}
\delta(C_{i_1}^*\cdots C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m+1-r}}^*) \sim\ & (2m-2r+1)\, C_{i_1}^*\cdots C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m-r}}^* G_{i_{2m+1-r}}\\
& - r\, C_{i_1}^*\cdots C_{i_{r-1}}^*\delta C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m+1-r}}^*.
\end{aligned}$$

The same reasoning as in the even case before gives us at odd antifield number $k = 2m+1$ the system

$$(X_r\, G_{i_{2m+1-r}} - X_{r+1}\, y_{i_{r+1}})\, C_{i_1}^*\cdots C_{i_r}^* h_{i_{r+1}}^*\cdots h_{i_{2m-r}}^* = \partial_\mu(\cdots)^\mu, \tag{7.72}$$

$$X_m\, C_{i_1}^*\cdots C_{i_m}^* G_{i_{m+1}} = \partial_\mu(\cdots)^\mu, \tag{7.73}$$

for $r = 0, \ldots, m-1$. Notice that Equations (7.70) and (7.72) are both of the form

$$(X_r\, G_{i_{r+t+1}} - X_{r+1} y_{i_{r+1}})\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} = \partial_\mu (\cdots)^\mu. \tag{7.74}$$

We shall want to argue, using the symmetry properties of X_r and X_{r+1} and the fact that the left summand vanishes on-shell, that this coupled equation actually splits into the following uncoupled equations

$$X_r\, G_{i_{r+t+1}} C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} = \partial_\mu (\cdots)^\mu, \tag{7.75}$$

$$X_{r+1} y_{i_{r+1}}\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} = \partial_\mu (\cdots)^\mu. \tag{7.76}$$

Consider for $r \geq 1$

$$\delta(X\, C^*_{i_1} \cdots C^*_{i_r} h^*_i) = X\, C^*_{i_1} \cdots C^*_{i_r} G_i - rX\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_i \tag{7.77}$$

and notice that the action of X in i is arbitrary. In the context of equation (7.65), this means that without loss of generality we may assume that the term $X_m\, C^*_{i_1} \cdots C^*_{i_m}$ is not in the differential ideal generated by the equations of motion, since, modulo δ-exact terms, any such term contained in the ideal, can simply be absorbed into the preceding term $X_{m-1}\, C^*_{i_1} \cdots C^*_{i_{m-1}} h^*_{i_m} h^*_{i_{m+1}}$. Analogously, for $t \geq 1$ the relation

$$\begin{aligned}\delta(X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t+1}}) = {} & (t+1) X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} G_{i_{r+t+1}} \\ & - rX\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t+1}}\end{aligned}$$

implies that in the context of equation (7.65), modulo δ-exact terms, terms of the form $X_r\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{2m-r}} G_{i_{2m-r+1}}$ may be absorbed into the X_{r-1} term, provided X_r is anti-symmetric in $i_{r+1}, \ldots, i_{2m-r+1}$. More generally, when considering the same term, but without any a priori specified symmetry property in the last index of the operator X_r, representation theory of the symmetric group tells us that we can split X_r into a sum

$$X_r[i_1, \ldots, i_{2m-r+1}] = X^a_r[i_1, \ldots, i_{2m-r+1}] + X^s_r[i_1, \ldots, i_{2m-r+1}], \tag{7.78}$$

where X^a_r is completely anti-symmetric in $i_{r+1}, \ldots, i_{2m-r+1}$ and X^s_r vanishes when performing the complete anti-symmetrization over $i_{r+1}, \ldots, i_{2m-r+1}$. Again, as before the X^a_r term can be absorbed into the X_{r-1} term. Continuing this argument, we see that the operators X_r are either not contained in the differential ideal generated by the equations of motion, or they vanish under the appropriate anti-symmetrization involving the anti-field indices and a single index associated to equations of motion.

Independent of the discussion so far, on-shell (in both even and odd antifield number) the systems (7.70) and (7.72), (7.73) reduce to solving the single case

$$X\, C^*_{i_1} \cdots C^*_{i_{r-1}} y_{i_r} h^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} \approx \partial_\mu (\cdots)^\mu, \tag{7.79}$$

where X is symmetric in $i_1, \ldots, i_r$ and anti-symmetric in $i_{r+1}, \ldots, i_{r+t}$, and according to Theorem 7.3.1 we may assume that the only h dependence (and hence the only x dependence) of X is through the Weyl-like tensor $\mathcal{W}$. Let us separate X according to field content:

$$X = \sum_{w=0}^{\infty} X_w, \tag{7.80}$$

where the index w describes the number of $\mathcal{W}$'s appearing in X_w, i.e. X_0 is constant, X_1 is linear in $\mathcal{W}$ and so on. It does not matter if this formal series terminates or not. Clearly, Equation (7.79) splits according to w-level and we get

$$X_w\, C^*_{i_1} \cdots C^*_{i_{r-1}} y_{i_r} h^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+t}} \approx \partial_\mu (\cdots)^\mu, \qquad \text{for } w = 0, 1, 2, \ldots. \tag{7.81}$$

Assuming we are able to show that, on-shell, the cohomology of δ modulo d is trivial, we see that we can separate the terms in Equation 7.64 into those that are contained in the ideal generated by the equations of motion, and those that are not. Both of those groups then have to be d-exact on their own. In the same spirit, we can further split those contained in the ideal into terms that vanish under the aforementioned anti-symmetrization and those that do not. It should be easy to show that both of those groups also have to be d-exact on their own.

Finally, if we are able to find suitable representatives for each of those d-exact terms, Equation 7.64 should reduce to, in the case of even antifield number $k = 2m$, solving the following uncoupled relations for the operators X_r

$$X_0\, h^*_{i_1} \cdots h^*_{i_{2m-1}} G_{i_{2m}} = 0, \tag{7.82}$$

$$X_r\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{2m-r-1}} G_{i_{2m-r}} = 0, \tag{7.83}$$

$$X_r y_{i_r}\, C^*_{i_1} \cdots C^*_{i_{r-1}} h^*_{i_r} \cdots h^*_{i_{2m-r}} = 0, \tag{7.84}$$

$$X_m y_{i_m}\, C^*_{i_1} \cdots C^*_{i_{m-1}} h^*_{i_m} = 0, \tag{7.85}$$

for $r = 1, \ldots, m-1$. Remember that the subscript of the operators X_r also serves to indicate their symmetry properties, i.e. X_r is symmetric in the first r indices appearing, and anti-symmetric in the rest. In the case of odd anti-field number $k = 2m+1$, we get, under consideration of the previous arguments, the relations

$$X_0\, h^*_{i_1} \cdots h^*_{i_{2m}} G_{i_{2m+1}} = 0, \tag{7.86}$$

$$X_r\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{2m-r}} G_{i_{2m+1-r}} = 0, \tag{7.87}$$

$$X_r y_{i_r}\, C^*_{i_1} \cdots C^*_{i_{r-1}} h^*_{i_r} \cdots h^*_{i_{2m+1-r}} = 0, \tag{7.88}$$

for $r = 1, \ldots, m$. We see that the cases of even antifield number and odd antifield number are essentially equivalent, with the exception that X_m has to satisfy only a single relation in the even case, cf. Equation (7.85). What follows is a rough argument why, in antifield number strictly greater than two, there are no nontrivial solutions to Equations (7.82) to (7.88).

1. We start with the case
$$\delta\big(X\, h^*_{i_1} \cdots h^*_{i_r}\big) = 0,$$
for $r \geq 2$, which corresponds to the cases (7.82) and (7.86). There are only two ways for this equation to hold. Either because of the change in symmetry due to
$$\delta \colon h_* \longmapsto G,$$
or via the Noether identities of G.

   (a) In the first case X has to be of the form $\widetilde{X}\, G_{i_0}$, i.e.
$$X\, h^*_{i_1} \cdots h^*_{i_r} = \widetilde{X}\, G_{i_0} h^*_{i_1} \cdots h^*_{i_r},$$

with the additional condition that the operator $\widetilde{X}$ has to be completely anti-symmetric in $i_0, \ldots, i_r$. But such a term is always in the image of δ because

$$\delta\big(\widetilde{X}\, h^*_{i_0} h^*_{i_1} \cdots h^*_{i_r}\big) = (r+1)\widetilde{X}\, G_{i_0} h^*_{i_1} \cdots h^*_{i_r}.$$

(b) In the latter case the Noether identities N have to act not only on one, but on each anti-field, because of the anti-symmetrization in the field indices, so X has to be of the form $\widetilde{X}\, N_{i_1} \cdots N_{i_r}$, i.e. more accurately

$$X\, h^*_{i_1} \cdots h^*_{i_r} = \widetilde{X}\, N_{i_1} h^*_{i_1} \cdots N_{i_r} h^*_{i_r},$$

with $\widetilde{X}$ anti-symmetric in $i_1, \ldots, i_r$, but this term is again in the image of δ, because

$$\widetilde{X}\, N_{i_1} h^*_{i_1} \cdots N_{i_r} h^*_{i_r} = \widetilde{X}\, \delta C^*_{i_1} \cdots \delta C^*_{i_r} = \frac{1}{r}\delta\big(\widetilde{X}\, C^*_{i_1} \delta C^*_{i_2} \cdots \delta C^*_{i_r}\big).$$

2. Next we consider the case

$$X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p-1}} G_{i_{r+p}} = 0, \tag{7.89}$$

$$X\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} = 0, \tag{7.90}$$

for $r \geq 1$ and $p \geq 2$. There are again two possibilities for the first condition to be fulfilled; either due to the change in symmetry or due to the Noether identities.

(a) Symmetry: $X = \widetilde{X} G_{i_{r+p+1}}$, with $\widetilde{X}$ anti-symmetric in i_{r+p} and i_{r+p+1}. Because

$$\begin{aligned}\delta\big(\widetilde{X}\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p+1}}\big) = (-1)^{p+1} r \widetilde{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p+1}} \\ + (p+1)\widetilde{X}\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} G_{i_{r+p+1}},\end{aligned}$$

we get up to δ-exact terms:

$$X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} = \widetilde{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p+1}},$$

where we also absorbed a numerical prefactor into $\widetilde{X}$. We also reverted to using δC^* instead of an explicit form, in order to emphasize the symmetry properties. Condition (7.89) now becomes

$$\begin{aligned}0 &= \delta\big(X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}}\big) \\ &= \delta\big(\widetilde{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p+1}}\big) \\ &= \widetilde{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} G_{i_{r+p+1}},\end{aligned}$$

where in the last line we ignored a numerical prefactor and used that $\widetilde{X}$ is symmetric in i_{r-1} and i_r, while the term $\delta C^*_{i_{r-1}} \delta C^*_{i_r}$ is anti-symmetric in these indices. Once again there are two possibilities for this condition to be fulfilled:

i. Symmetry: $\widetilde{X} = \widehat{X} G_{i_{r+p+2}}$, with $\widehat{X}$ anti-symmetric in i_{r+p+1} and i_{r+p+2}. Notice that the term

$$\widehat{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p+1}} G_{i_{r+p+2}}$$

is already exact, because it can be written as

$$\frac{1}{p+1}\, \delta\big(\widehat{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h_{i_{r+p+2}}\big).$$

ii. Noether: $\widetilde{X} = \widehat{X} N_{i_{r+1}} \cdots N_{i_{r+p+1}}$, where $\widehat{X}$ has the same symmetry properties as $\widetilde{X}$. But again, the term

$$\widehat{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} N_{i_{r+1}} h^*_{i_{r+1}} \cdots N_{i_{r+p+1}} h^*_{i_{r+p+1}}$$

is already exact, because it can be written as

$$\frac{1}{r}\, \delta\big(\widehat{X}\, C^*_{i_1} \cdots C^*_{i_r} \delta C^*_{i_{r+1}} \cdots \delta C^*_{i_{r+p+1}}\big).$$

(b) Noether: $X = \widetilde{X} N_{i_{r+1}} \cdots N_{i_{r+p}}$, where $\widetilde{X}$ has the same symmetry properties as X. So, again as before, the second condition

$$X\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} = 0,$$

becomes

$$\widetilde{X}\, C^*_{i_1} \cdots C^*_{i_{r-1}} \delta C^*_{i_r} \delta C^*_{i_{r+1}} \cdots \delta C^*_{i_{r+p}} = 0,$$

which can only be satisfied if $\widetilde{X}$ is symmetric in i_r, i_{r+1}, which now implies that

$$X\, C^*_{i_1} \cdots C^*_{i_r} h^*_{i_{r+1}} \cdots h^*_{i_{r+p}} = \delta\big(\widetilde{X}\, C^*_{i_1} \cdots C^*_{i_r} C^*_{i_{r+1}} \delta C^*_{i_{r+2}} \cdots \delta C^*_{i_{r+p}}\big).$$

3. Finally we consider the case

$$\delta\big(X\, C^*_{i_1} \cdots C^*_{i_r}\big) = 0,$$

for $r \geq 2$. Because δC^* does not satisfy any relations, which do not also hold already for C^*, the only possibility for this case to be satisfied is due to the change in symmetry. Just as in the case 1(a), a term with the appropriate symmetry properties is automatically δ-exact.

From this point forward, one should be able to generalize the constructions given in [BBC06]. Recall from Section 6.2, that local, first order, non-trivial, consistent interactions correspond exactly to the distinct elements of the homology group $H^{D,0}(s|d)$. When expanding a representative of an element $[a] \in H^{D,0}(s|d)$ according to antifield number

$$a = a_0 + a_1 + \cdots + a_m, \tag{7.91}$$

we showed in Section 7.4 that, if the cohomology groups $\mathrm{H}_k^{D,\mathrm{inv}}(\delta|d)$ vanish for $k > 2$, we can assume without loss of generality that the expansion (7.91) stops at antighost number two, i.e. it is of the form

$$a = a_0 + a_1 + a_2. \tag{7.92}$$

Similar to the construction of the descends in the previous section, the defining equation $sa + db = 0$ yields, writing $b = b_0 + b_1$ and $s = \gamma + \delta$, the system

$$\gamma a_0 + \delta a_1 + d b_0 = 0, \tag{7.93}$$

$$\gamma a_1 + \delta a_2 + d b_1 = 0, \tag{7.94}$$

$$\gamma a_2 = 0. \tag{7.95}$$

Using Theorem 7.3.1 and decomposing the last equation according to d'-degree, we should be able to list all of its solutions a_2. Then, after finding the solutions to $\gamma a_2 = 0$ one still has to check that there exist solutions to Equations (7.93) and (7.94). We suspect that these equations can be solved by explicitly constructing particular solutions and giving the general solution to the homogeneous problem $\gamma a_i = 0$, which is equivalent to $\gamma a_i = d b_i$, as we showed in Theorem 7.3.6.

Chapter 8

Conclusion

In this work, we have given a rough argument as to why, in higher-spin gauge theories of arbitrary integer spin, the cohomology groups $H_k^{D,\mathrm{inv}}(\delta|d)$ vanish for antifield number k strictly greater than two. The argument is not yet rigorous, but we believe that it can be made so, once the indicated steps towards the end of the last chapter have been fleshed out.

The main problem, which we have been unable to solve as yet, is related to the fact that the kernel of the Koszul-Tate derivative has to be computed in terms of equivalence classes modulo d-exact terms. In other words, when discussing its action on representatives, the kernel of the Koszul-Tate derivative is actually the preimage of all total derivatives at the appropriate antifield number. This inconspicuous detail lies at the heart of most of the complications we have faced.

We are quite confident that the arguments laid out in this work are sufficient to be able to separate the computation of the kernel of the Koszul-Tate derivative according to antifield and antighost content. At that point, one would like to choose, for each relevant term, a convenient representative which makes it clear that its image under the Koszul-Tate derivative can only be a total derivative if the original term also shares this property, but due to the symmetry properties of the antifields and antighosts, we have not had much success in this direction.

We believe a better characterization of those terms which can be written as total derivatives is necessary in order to continue. Once this problem has been resolved there remains the small issue of the relations between the Fronsdal tensors and curvature tensors. We expect these relations not to influence the computation of the relevant cohomology groups, but again due to the issue of total derivatives, we have not been able to give a convincing argument so far.

BIBLIOGRAPHY

[And92] I. Anderson. "Introduction to the Variational Bicomplex". In: *Contemporary Mathematics* 132 (Jan. 1, 1992).

[BB03] X. Bekaert and N. Boulanger. *Mixed Symmetry Gauge Fields in a Flat Background*. Dec. 2, 2003. arXiv: hep-th/0310209.

[BB04] X. Bekaert and N. Boulanger. "Tensor Gauge Fields in Arbitrary Representations of GL(D,R) : Duality & Poincare Lemma". In: *Communications in Mathematical Physics* 245.1 (Feb. 1, 2004), pp. 27–67. arXiv: hep-th/0208058.

[BB05] X. Bekaert and N. Boulanger. "Gauge Invariants and Killing Tensors in Higher-Spin Gauge Theories". In: *Nuclear Physics B* 722.3 (Aug. 2005), pp. 225–248. arXiv: hep-th/0505068.

[BBC06] X. Bekaert, N. Boulanger, and S. Cnockaert. "Spin Three Gauge Theory Revisited". In: *Journal of High Energy Physics* 2006.01 (Jan. 13, 2006), pp. 052–052. arXiv: hep-th/0508048.

[BBH00] G. Barnich, F. Brandt, and M. Henneaux. "Local BRST Cohomology in Gauge Theories". In: *Physics Reports* 338.5 (Nov. 2000), pp. 439–569. arXiv: hep-th/0002245.

[BBH95a] G. Barnich, F. Brandt, and M. Henneaux. "Local BRST Cohomology in the Antifield Formalism: I. General Theorems". In: *Communications in Mathematical Physics* 174.1 (Nov. 1995), pp. 57–91. arXiv: hep-th/9405109.

[BBH95b] G. Barnich, F. Brandt, and M. Henneaux. "Local BRST Cohomology in the Antifield Formalism: II. Application to Yang-Mills Theory". In: *Commun. Math. Phys.* 174.1 (Nov. 1995), pp. 93–116. arXiv: hep-th/9405194.

[BBL10] X. Bekaert, N. Boulanger, and S. Leclercq. "Strong Obstruction of the Berends –Burgers–van Dam Spin-3 Vertex". In: *Journal of Physics A: Mathematical and Theoretical* 43.18 (May 7, 2010), p. 185401.

[BBvD84] F. A. Berends, G. J. H. Burgers, and H. van Dam. "On Spin Three Self Interactions". In: *Zeitschrift für Physik C Particles and Fields* 24.3 (Sept. 1, 1984), pp. 247–254.

[BD18] G. Barnich and F. Del Monte. *Introduction to Classical Gauge Field Theory and to Batalin-Vilkovisky Quantization*. Sept. 30, 2018. arXiv: 1810 . 00442 [hep-th].

[Bek+20] X. Bekaert et al. *Nonlinear Higher Spin Theories in Various Dimensions*. June 12, 2020. arXiv: hep-th/0503128.

[BH93] G. Barnich and M. Henneaux. "Consistent Couplings between Fields with a Gauge Freedom and Deformations of the Master Equation". Version 1. In: *Physics Letters B* 311.1-4 (July 1993), pp. 123–129. arXiv: hep-th/9304057.

[BLS08] N. Boulanger, S. Leclercq, and P. Sundell. "On The Uniqueness of Minimal Coupling in Higher-Spin Gauge Theory". In: *Journal of High Energy Physics* 2008.08 (Aug. 19, 2008), pp. 056–056. arXiv: 0805.2764.

[Bou+01] N. Boulanger et al. "Inconsistency of Interacting, Multi-Graviton Theories". In: *Nuclear Physics B* 597.1-3 (Mar. 2001), pp. 127–171. arXiv: hep-th/0007220.

[BRS75] C. Becchi, A. Rouet, and R. Stora. "Renormalization of the Abelian Higgs-Kibble Model". In: *Communications in Mathematical Physics* 42.2 (June 1, 1975), pp. 127–162.

[BV81] I. Batalin and G. Vilkovisky. "Gauge Algebra and Quantization". In: *Physics Letters B* 102.1 (June 1981), pp. 27–31.

[BV83] I. A. Batalin and G. A. Vilkovisky. "Quantization of Gauge Theories with Linearly Dependent Generators". In: *Physical Review D* 28.10 (Nov. 15, 1983), pp. 2567–2582.

[CP19] J. C. Criado and M. Perez-Victoria. "Field Redefinitions in Effective Theories at Higher Orders". In: *Journal of High Energy Physics* 2019.3 (Mar. 2019), p. 38. arXiv: 1811.09413.

[DH02] M. Dubois-Violette and M. Henneaux. "Tensor Fields of Mixed Young Symmetry Type and N-Complexes". In: *Communications in Mathematical Physics* 226.2 (Mar. 1, 2002), pp. 393–418. arXiv: math/0110088.

[DH99] M. Dubois-Violette and M. Henneaux. *Generalized Cohomology for Irreducible Tensor Fields of Mixed Young Symmetry Type*. July 22, 1999. arXiv: math/9907135.

[Fis+89] J. Fisch et al. "Existence, Uniqueness and Cohomology of the Classical BRST Charge with Ghosts of Ghosts". In: *Communications in Mathematical Physics* 120.3 (Sept. 1989), pp. 379–407.

[FKM20] S. Fredenhagen, O. Krüger, and K. Mkrtchyan. "Restrictions for N-Point Vertices in Higher-Spin Theories". In: *Journal of High Energy Physics* 2020.6 (June 18, 2020), p. 118.

[Fro78] C. Fronsdal. "Massless Fields with Integer Spin". In: *Physical Review D* 18.10 (Nov. 15, 1978), pp. 3624–3629.

[FS02] D. Francia and A. Sagnotti. "Free Geometric Equations for Higher Spins". In: *Physics Letters B* 543.3 (Sept. 12, 2002), pp. 303–310.

[GMS09] G. Giachetta, L. Mangiarotti, and G. Sardanashvily. *Advanced Classical Field Theory*. WORLD SCIENTIFIC, May 2009.

[Hen90a] M. Henneaux. "Elimination of the Auxiliary Fields in the Antifield Formalism". In: *Physics Letters B* 238.2-4 (Apr. 1990), pp. 299–304.

[Hen90b] M. Henneaux. "Lectures on the Antifield-BRST Formalism for Gauge Theories". In: *Nuclear Physics B - Proceedings Supplements* 18.1 (Dec. 1, 1990), pp. 47–105.

[HT92] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton, N.J: Princeton University, 1992. 520 pp.

[MMS02] E. Martinez, T. Mestdag, and W. Sarlet. “Lie Algebroid Structures and Lagrangian Systems on Affine Bundles”. In: *Journal of Geometry and Physics* 44.1 (Nov. 2002), pp. 70–95. arXiv: `math/0203178`.

[NO90] N. Nakanishi and I. Ojima. *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*. World Scientific Lecture Notes in Physics v. 27. Singapore ; Teaneck, NJ: World Scientific, 1990. 434 pp.

[Sau89] D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge: Cambridge University Press, 1989.

[Sor05] D. Sorokin. “Introduction to the Classical Theory of Higher Spins”. In: *AIP Conference Proceedings* 767 (2005), pp. 172–202. arXiv: `hep-th/0405069`.

[Sta97] J. Stasheff. *The (Secret?) Homological Algebra of the Batalin-Vilkovisky Approach*. Dec. 16, 1997. arXiv: `hep-th/9712157`.

[Tyu75] I. V. Tyutin. “Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism”. In: *Lebedev preprint FIAN* 39 (1975). arXiv: `0812.0580v2`.